Semi-separability and conditions up to retracts*

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In [1] we have introduced a weakening of the notions of separable [7] and naturally full [2] functors, that we have called *semi-separable*. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *separable* if the associated natural transformation $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F-,F-)$, mapping f to $F(f)$, has a left inverse, i.e. there is a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-,F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(-,-)}$, whereas a *naturally full* functor is defined to satisfy a right version of this property. Semi-separable functors result to be relevant to provide novel characterizations of separable and naturally full functors. In this talk, we explore semi-separability in connection with functors admitting a fully faithful left or right adjoint, called *coreflections* and *reflections* [3], respectively. Being a coreflection (resp. reflection) is equivalent to the fact that the unit (resp. counit) of the corresponding adjunction is an isomorphism. Inspired by these notions and from [5], we say that a functor is a *bireflection* if it has a left and right adjoint equal, which is fully faithful and satisfies a coherence condition relating the unit and counit of the adjunctions. After providing a characterization of bireflections in terms of semi-separable (co)reflections, we introduce and investigate the notions of *(co)reflection up to retracts* and *bireflection up to retracts*, which do not depend on the existence of an adjoint, in parallel with the notion of *equivalence up to retracts* [4] assigned to a functor $F : \mathcal{C} \to \mathcal{D}$ whose completion $F^\natural : \mathcal{C}^\natural \to \mathcal{D}^\natural$ is an equivalence. We show that the canonical quotient functor $H : \mathcal{C} \to \mathcal{C}_e$, associated to the coidentifier category $\mathcal{C}_e$ [5] attached to a category $\mathcal{C}$ and to an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, is a coreflection up to retracts. Moreover, it will result to be a bireflection up to retracts. Given an adjunction $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$, with unit η and counit ϵ, we can consider the Eilenberg-Moore categories $\mathcal{C}_{GF}$ and $\mathcal{D}^{FG}$, of modules over the monad $(GF, G\epsilon F, \eta)$ and of comodules over the comonad $(FG, F\eta G, \epsilon)$, respectively. When the right adjoint G is semi-separable, we prove that the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$, $K_{GF}D = (GD, G\epsilon_D)$, $K_{GF}f = Gf$, is a bireflection up to retracts, and analogously so is the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$, $K^{FG}C = (FC, F\eta_C)$, $K^{FG}f = Ff$, when the left adjoint F is semi-separable. Furthermore, the semi-separability of the right adjoint G provides an equivalence between the associated Kleisli category $GF\text{-Free}_{\mathcal{C}}$ of free GF-modules [6] and $\mathcal{C}_{GF}$, after idempotent completion. As a consequence, these categories are also equivalent up to retracts to the coidentifier category associated to the semi-separable right adjoint. This is based on a joint work in progress with Alessandro Ardizzoni (University of Turin).

References

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*Joint work with Alessandro Ardizzoni (University of Turin). Abstract submitted to ItaCa 2021 Workshop.