Doctrines, modalities and comonads

Francesco Dagnino
joint work with Giuseppe Rosolini

The approach to logic proposed by F.W. Lawvere via hyperdoctrines has proved very fruitful as it provides an extremely suitable environment where to analyse both syntacic aspects of logic and semantic aspects as well as compare one with the other, see [2, 3]. The suggestion is to see a logic as a functor $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ from the opposite of a category to the category of posets and monotone functions where the category $\mathcal{C}$ collects the "types" of the logic and terms in context, a poset $P(c)$ presents the "properties" of the type c with the order relation describing their "entailments". Since they are structured categories, doctrines get swiftly organised in a 2-category, which thus provides a unifying environment to study logic.

One of the main points of Lawvere's structural approach to logic is that all the logical operators are obtained from adjunctions. That view in itself is very powerful and contributes to unifying many different aspects in logic. In this talk, we show that also a wide class of modal operators, namely, those satisfying axioms T and 4, is obtained from adjunctions.

Typically, modalities are unary logical operators, which are quite well-understood in the context of propositional logic. However, their meaning is less clear in a typed logical formalism. In this setting, there are various semantics which are interrelated, and we show that many of these are instances of the general situation of an adjunction between two homomorphisms of doctrines.

We show that an adjunction in the 2-category of doctrines gives rise to a doctrine with a modal operator. Such an adjunction is very much like an adjunction between categories: roughly, it consists of two doctrines $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ and $Q\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{P}os$ and two homomorphisms of doctrines connecting them, which should be thought of as an interpretation of P in Q (the left adjoint) and an interpretation of Q in P (the right adjoint). Such a situation can be summarised by a modal logic which uses the logic Q to describe properties of types in $\mathcal{C}$ (the base category of P) and the modal operator to recover (an image of) properties described by P. In a sense, we extend the logic P through the adjunction to a richer logic and use a modal operator to keep memory of the original logic. As we said, many standard approaches to the semantics of modal logic are instances of such construction.

Taking a slightly different perspective, we show that also a comonad in the 2-category of doctrines determines a doctrine with a modal operator, this time on the category of coalgebras for the comonad. Intuitively, we get a logic where types have a dynamics, given by the coalgebra structure, and the modal operator specifies when a property is invariant for such dynamics.

These two constructions are tightly related. Relying on results in [1], we show that every comonad in the 2-category of doctrines determines an adjunction, hence, also a modal operator. In fact, the construction starting from comonads is defined in this way. On the other hand, every adjunction determins a comonad, hence a modal operator. However, the two construction starting from an adjunction do not coincide, but we show they can be canonically compared by a homomorphism of doctrines preserving the modal operator. FInally, we measure in a categorical form how the passage to a modal operator hides part of the structure that generated it.

References

[1] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *J. Pure Appl. Algebra*, 59(1):1–41, 1989.

[2] F. W. Lawvere. Adjointness in foundations. *Dialectica*, 23:281–296, 1969. also available as Repr. Theory Appl. Categ., 16 (2006) 1–16.

[3] F. W. Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In A. Heller, editor, *Proc. New York Symposium on Application of Categorical Algebra*, pages 1–14. Amer.Math.Soc., 1970.