GIVANT, MORLEY, ZILBER

BITTER REMEDIES FOR ROTA'S CURSE.

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> The elements of category theory are presented with unsurpassed clarity and full motivation [...]. One or two more books like this one and universal algebra might take off.
>
> Gian-Carlo Rota on Toposes, Triples and Theories [1], 1986.

For a unital commutative ring R it is well established that R is a field if and only iff all its modules are free. From the point of view of a universal algebraist, this result shows how structural properties of an algebraic theory are tightly related to the behavior of its variety of algebras. As a general pattern, it is as fascinating as unsurprising: similar results are known for semisimple artinian rings, and somewhat related classes of rings. In these cases coming from every-day-life-algebra, it is fair to say that the proof techniques are category theoretic, or at least digested by the category theoretic language.

Yet, when one raises the question *can we characterize the finitary monads such that every algebra is free?*, we suddenly realize the complexity of the situation, and the most updated category theoretic techniques apparently offer no explanation for the phenomenon. The question was raised several times, [6, 7], always with similar spirit. In 1975, Givant [2] provided a full characterization of those varieties for which every algebra is free, those are: sets, pointed sets, vector spaces over a division ring or affine spaces over a division ring. We have no categorical understanding of why this is true.

One should ask whether the question is a good one in first place, maybe the reason for which there are so few examples is a mere coincides. We will show during the talk that this is not the case, hinting at some very geometric behaviors that put some constraints on the monad. For example, in the case of vector spaces, the result is based on: *the extraction of a basis, the notion of span, linear independence.* We look at this *Givant programme* the same way the community of model theory looks at Morley's categoricy theorem (the two problems are indeed related) [4]. While it is true that the answer per se is not that interesting, the techniques that are needed in order to settle its status have an incredible importance in our understanding of Universal Algebra.

Notions of basis, linear independence and span can be defined via closure operators for every monad on the category of sets, and there is evidence that these ideas can be applied successfully to the characterization of free objects. Indeed, the whole program of geometric stability theory, initiated by Zilber [3], uses these techniques to solve similar problems, as shown by Hyttinen and Kangas [5].

In 2002, Lawvere suggested on the catlist the Boone conjecture as a test problem for the maturity of the technical apparatus of Categorical Logic. We feel similarly about this question, but we believe that some new ideas will be needed, and there are many things to be learned from the techniques of Givant, Zilber and Morley

that could enrich Categorical Logic. Those ideas, when correctly generalized, could be exported to other contexts, for example monads withour arity.

This talk is not based on any preprint, nor work in progress, it is instead an inspirational account on a collection of topics and traditions that are scattered among the communities interested in universal algebra, model theory, algebraic geometry and related areas.

References

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[3] Zilber Boris, *Hereditarily Transitive Groups and Quasi-Urbanik Structures*, Model Theory and Applications. AMS Translations 2, Vol 195 (1999).

[4] Morley Michael, *Categoricity in power*, Model Theory and Applications. Transactions of the American Mathematical Society 114.2 (1965): 514-538.

[5] Hyttinen, Tapani, and Kaisa Kangas, *Categoricity and universal classes*, Mathematical Logic Quarterly 64.6 (2018): 464-477.

[6] Tim Campion, *Varieties where every algebra is free*, MathOverflow.

[7] Tim Campion, *Varieties where every algebra is projective?*, MathOverflow.