Applications of (weak) generalized existential completions to regular and exact Morita-equivalences

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We introduce the notion of *regular Morita-equivalence* and *exact Morita-equivalence* for elementary existential doctrines, motivated by the notion of Morita equivalence between regular theories introduced in [4]. In more details, we say that two elementary and existential doctrines are regular Morita-equivalent/exact Morita-equivalent when their regular/exact completions [7, 6] are equivalent. Employing these notions together with the characterizations of doctrines arising as full existential completions presented in [5] we show that the regular completion $\mathsf{Reg}(P)$ of an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is an instance of the reg/lex completion [2] *if and only* if P is regular Morita-equivalent to a full existential completion, while $\mathsf{Reg}(P)$ is an instance of reg/wlex completion *if and only if* P is regular Morita-equivalent to a *weakly full existential completion.*

Then we show that the exact completion $\mathcal{T}_P$ of an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is an instance of ex/lex completion *if and only if* P is exact Morita-equivalent to a full existential completion, while $\mathcal{T}_P$ is an instance of ex/wlex completion *if and only if* P is exact Morita-equivalent to a *weakly full existential completion.* Finally we present relevant applications of these results to tripos-to-topos constructions including all realizability toposes [3] and localic toposes associated to a supercoherent locale [1], to regular and exact completions of syntactic doctrines and to the construction of Joyal's arithmetic universes.

References

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