A topos-theoretic perspective on properly discontinuous actions

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Among the first examples students of topos theory are introduced to is the topos of sheaves on a space X with an action by a group G, denoted by G-**Étale**$/X$. This is the category whose objects are local homeomorphisms $Y \xrightarrow{q} X$ where Y has a G-action and q commutes with this action.

It is also presented early on, as an exercise in both Johnstone's *Topos Theory* ([3, Exercise 0.9]) and Mac Lane and Moerdijk's *Sheaves in Geometry and Logic* ([4, Exercise II.10]), that whenever the group action is *properly discontinuous* (proper in the terminology of Mac Lane and Moerdijk) the topos G-**Étale**$/X$ is localic. That is, if there is a covering of X by opens U_i such that, for all $g \in G$,

$$gU_i \cap U_i \neq \emptyset \implies g = e,$$

then there is an equivalence $G\text{-}\mathbf{\acute{E}tale}/X \simeq \mathbf{\acute{E}tale}/(X/G)$. In [3, Exercise 0.9], Johnstone also sets, as an extension, that if the G action is *free* (if $g \cdot x = x$ implies that $g = e$) then the converse holds as well.

In this paper, we will do two things.

1. We will reprove the results found in [3, Exercise 0.9] and [4, Exercise II.10], but this time without the need to assume freeness of the action for the converse. Thus, we can conclude that properness of the action is a site-level description of a topos theoretic invariant (the topos G-**Étale**$/X$ being localic) in the style of Caramello's bridge technique (cf. [1, §2.2]).
2. Secondly, we will expand the techniques employed in the discrete case to the extended problem where G is a topological group and all G-actions are required to be continuous. Thereby, we obtain a necessary and sufficient condition for the topos G-**Étale**$/X$ to be localic. Properness is then recovered as a special case for discrete groups.

The method employed involves considering the functor $b\colon \mathcal{E} \to \mathrm{Sub}_{\mathcal{E}}(1)$ that sends an object $E \in \mathcal{E}$ to the subterminal given by the epi-mono factorisation of the unique morphism $E \twoheadrightarrow b(E) \rightarrowtail 1$ to the terminal object. The functor b forms a comorphism of sites (cf. [2, §3.3]) $(\mathcal{E}, J_{can}) \to (\mathrm{Sub}_{\mathcal{E}}(1), J_{can})$. Therefore, so does the composite

$$\mathcal{C} \xrightarrow{\ \ell\ } \mathbf{Sh}(\mathcal{C}, J) \xrightarrow{\ b\ } \mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(1),$$

where ℓ is the canonical functor: the Yoneda embedding よ followed by sheafification a_J. We can then apply the theory developed in Section 7 of [2] to first show that the induced geometric morphism $C_{b\circ\ell}\colon \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathrm{Sub}_{\mathbf{Sh}(\mathcal{C},J)}(1))$ is hyperconnected, and therefore the canonical functor to the localic reflection. Then, via [2, Proposition 7.11], we can find necessary and sufficient conditions for $C_{b\circ\ell}$ to be an equivalence. Using a modification of the Moerdijk site for the topos of sheaves on a general groupoid found in [5, §6], a computationally feasible condition is found for the topos G-**Étale**$/X$, which we can then compare to the previously known results such as those found in [6, Chapter 8].

It is hoped this paper will be of interest not only as a study of a special case of Morita equivalence between two topological groupoids, but also as a pratical application of the techniques for comorphisms of sites presented in [2, §7]. This paper represents ongoing work.

References

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