ItaCa Workshop 2021

Semi-separability and Conditions up to retracts

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Based on a joint work in progress with Alessandro Ardizzoni (University of Turin).

Genoa, 21 December 2021

Outline

1. Semi-separable functors
2. Semi-separability and (Co)reflections
3. Conditions up to retracts
4. Eilenberg-Moore and Kleisli categories

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1. Semi-separable functors
2. Semi-separability and (Co)reflections
3. Conditions up to retracts
4. Eilenberg-Moore and Kleisli categories

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Consider the associated natural transformation

$$\mathcal{F}^F : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F-,F-), \quad \mathcal{F}^F_{X,Y}(f) = F(f).$$

If there exists a natural transformation

$$\mathcal{P}^F : \mathrm{Hom}_{\mathcal{D}}(F-,F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$$

such that

- $\mathcal{P}^F \circ \mathcal{F}^F = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(-,-)}$, F is called separable;[1]
- $\mathcal{F}^F \circ \mathcal{P}^F = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{D}}(F-,F-)}$, F is called naturally full;
- $\mathcal{F}^F \circ \mathcal{P}^F \circ \mathcal{F}^F = \mathcal{F}^F$, we say that F is semi-separable.

1

Năstăsescu C., Van den Bergh M., Van Oystaeyen F., Separable functors applied to graded rings., *J. Algebra 123 (2)*, 1989, 397–413.

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- $\mathcal{F}^F \circ \mathcal{P}^F \circ \mathcal{F}^F = \mathcal{F}^F$, we say that F is semi-separable.

Properties:

- F is separable $\Leftrightarrow$ F is semi-separable and faithful
- F is naturally full $\Leftrightarrow$ F is semi-separable and full

[1]

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- Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors.
  F is semi-separable and G is separable $\Rightarrow$ $G \circ F$ is semi-separable
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- The *associated idempotent*: $F : \mathcal{C} \to \mathcal{D}$ is semi-separable $\Rightarrow \exists!$ idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ such that $Fe = \mathrm{Id}_F$ with the universal property: if $f, g : A \to B$ are morphisms, then

$$Ff = Fg \quad \Leftrightarrow \quad e_B \circ f = e_B \circ g.$$

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Corollary

Let $F : \mathcal{C} \to \mathcal{D}$ be a semi-separable functor and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be the associated idempotent natural transformation. Then,

F is separable $\Leftrightarrow e = \mathrm{Id}.$

Corollary

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then,

F is separable $\Leftrightarrow$ F is semi-separable and Maschke, i.e. it reflects split-monomorphisms.

The Coidentifier Category (Freyd et al., 1999)

Let $\mathcal{C}$ be a category and $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ an idempotent natural transformation. The coidentifier category $\mathcal{C}_e$ is given by

$$\mathrm{Ob}\,(\mathcal{C}_e) = \mathrm{Ob}\,(\mathcal{C}), \quad \mathrm{Hom}_{\mathcal{C}_e}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)/\!\sim, \text{ where } f \sim g \Leftrightarrow e_B \circ f = e_B \circ g$$

$\Rightarrow$ the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$, acting as the identity on objects and as the canonical projection on morphisms, is naturally full w.r.t. $\mathcal{P}^H_{A,B} : \mathrm{Hom}_{\mathcal{C}_e}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\mathcal{P}^H_{A,B}(\overline{f}) = e_B \circ f$, where $\overline{f}$ is the class of $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ in $\mathrm{Hom}_{\mathcal{C}_e}(A, B)$.

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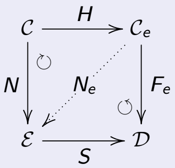
Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a semi-separable functor and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be the associated idempotent natural transformation. Then $\exists!$ $F_e : \mathcal{C}_e \to \mathcal{D}$, necessarily separable, such that $F = F_e \circ H$, where $H : \mathcal{C} \to \mathcal{C}_e$ is the canonical naturally full functor. Furthermore, if F also factors as $S \circ N$ where $S : \mathcal{E} \to \mathcal{D}$ is separable and $N : \mathcal{C} \to \mathcal{E}$ is naturally full, then



$\exists!$ $N_e : \mathcal{C}_e \to \mathcal{E}$, necessarily fully faithful, such that $N_e \circ H = N$ and $S \circ N_e = F_e$.

$\Rightarrow$ *A functor is semi-separable $\Leftrightarrow$ it factors as $S \circ N$ where S is a separable functor and N is a naturally full functor.*

Rafael Theorem, 1990

Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ.

Rafael M.D., Separable functors revisited. *Comm. Algebra 18*, 1990, 1445–1459.

1. F is separable $\Leftrightarrow \eta : \mathrm{Id}_{\mathcal{C}} \to GF$ splits, i.e. $\exists\, \nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$;
2. G is separable $\Leftrightarrow \epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ cosplits, i.e. $\exists\, \gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma = \mathrm{Id}_{\mathrm{Id}_{\mathcal{D}}}$.

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Example: $\varphi^* \dashv \varphi_*$

Let $\varphi : R \to S$ be a ring morphism, and consider the *extension* φ^* and the *restriction* of scalars functor φ_*:

$$R\text{-Mod} \underset{\varphi_*}{\overset{\varphi^* = S \otimes_R (-)}{\rightleftarrows}} S\text{-Mod} \qquad (\perp)$$

- φ^* is separable $\Leftrightarrow \exists\, \chi \in {}_R\mathrm{Hom}(S,R)_R$ such that $\chi \circ \varphi = \mathrm{Id}_R$.
- φ_* is separable $\Leftrightarrow S/R$ is separable, i.e. the product map $m_S : S \otimes_R S \to S$ has an S-bimodule section σ ($m_S \sigma = \mathrm{Id}_S$).
- φ^* is naturally full $\Leftrightarrow \exists\, \chi \in {}_R\mathrm{Hom}(S,R)_R$ such that $\varphi \circ \chi = \mathrm{Id}_S$.
- φ_* is naturally full $\Leftrightarrow$ it is full.

Rafael-type Theorem

Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then:

- F is semi-separable $\Leftrightarrow$ $\exists$ a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that one of the following equivalent conditions holds:
  - $\eta \circ \nu \circ \eta = \eta$;
  - $F\nu \circ F\eta = \mathrm{Id}_F$;
  - $\nu G \circ \eta G = \mathrm{Id}_G$.
- G is semi-separable $\Leftrightarrow$ $\exists$ a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that one of the following equivalent conditions holds:
  - $\epsilon \circ \gamma \circ \epsilon = \epsilon$;
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- G is semi-separable $\Leftrightarrow \exists$ a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that one of the following equivalent conditions holds:
  - $\epsilon \circ \gamma \circ \epsilon = \epsilon$;
  - $G\epsilon \circ G\gamma = \mathrm{Id}_G$;
  - $\epsilon F \circ \gamma F = \mathrm{Id}_F$.

Corollary

Let $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$ be an adjoint triple. Then, F is semi-separable (resp. separable, naturally full) $\Leftrightarrow$ so is H.

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Corollary

- If G is semi-separable, then the associated monad $(GF, G\epsilon F, \eta)$ is separable[a].
- If F is semi-separable, then the associated comonad $(FG, F\eta G, \epsilon)$ is coseparable.

[a] A monad $(\top, m : \top\top \to \top, \eta : \mathrm{Id}_{\mathcal{C}} \to \top)$ on a category $\mathcal{C}$ is *separable* if $\exists\ \sigma : \top \to \top\top$ such that $m \circ \sigma = \mathrm{Id}_{\top}$ and $\top m \circ \sigma \top = \sigma \circ m = m\top \circ \top\sigma$. A *coseparable* comonad is defined dually.

If G (resp. F) is separable, cf. [Chen X.-W., A note on separable functors and monads with an application to equivariant derived categories. *Abh. Math. Semin. Univ. Hambg. 85* (2015), no. 1, 43–52.]

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2. Semi-separability and (Co)reflections

3. Conditions up to retracts

4. Eilenberg-Moore and Kleisli categories

Recall that

- a functor admitting a fully faithful left adjoint is called a coreflection;
- a functor admitting a fully faithful right adjoint is called a reflection; see

BERGER C., Iterated wreath product of the simplex category and iterated loop spaces. *Adv. Math. 213*, 2007, no. 1, 230–270.

- a functor $G : \mathcal{D} \to \mathcal{C}$ is called a bireflection if it has a left and right adjoint equal, $F : \mathcal{C} \to \mathcal{D}$, which is fully faithful and satisfies a coherence condition $\gamma \circ \epsilon = \mathrm{Id}$, where $\epsilon : FG \to \mathrm{Id}$ is the counit of $F \dashv G$ and $\gamma : \mathrm{Id} \to FG$ is the unit of $G \dashv F$, cf.

FREYD P. J., O'HEARN P. W., POWER A. J., STREET R., TAKEYAMA M., TENNENT R. D., Bireflectivity. *Mathematical foundations of programming semantics (New Orleans, LA, 1995). Theoret. Comput. Sci. 228*, 1999, no. 1-2, 49–76.

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Remark

$G : \mathcal{D} \to \mathcal{C}$ is an equivalence $\Leftrightarrow$ it is a fully faithful bireflection.

AIM: to study semi-separable (co)reflections.

Semi-separable (co)reflections

The following are equivalent for a functor $G : \mathcal{D} \to \mathcal{C}$.

1) G is naturally full coreflection.
2) G is semi-separable coreflection.
3) G is a bireflection.
4) G is Frobenius coreflection.
5) G is naturally full reflection.
6) G is semi-separable reflection.
7) G is Frobenius reflection.

Semi-separable (co)reflections

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Sketch Proof.

Assume G is a coreflection. Denote by F the left adjoint of G, by $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ the unit and counit of the adjunction (F, G). Since F is fully faithful, then η is invertible. From $\epsilon F \circ F\eta = \mathrm{Id}_F$ and $G\epsilon \circ \eta G = \mathrm{Id}_G$, we get $(F\eta)^{-1} = \epsilon F$ and $(\eta G)^{-1} = G\epsilon$.
(2) $\Rightarrow$ (3). If G is semi-separable, then by Rafael-type Theorem there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$, i.e. such that $\epsilon F \circ \gamma F = \mathrm{Id}_F$ and $G\epsilon \circ G\gamma = \mathrm{Id}_G$, so

$$\begin{aligned} F\left(\eta^{-1}\right) \circ \gamma F &= (F\eta)^{-1} \circ \gamma F = \epsilon F \circ \gamma F = \mathrm{Id}_F, \\ \eta^{-1} G \circ G\gamma &= (\eta G)^{-1} \circ G\gamma = G\epsilon \circ G\gamma = \mathrm{Id}_G. \end{aligned}$$

This means that (G, F) is an adjunction with unit $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ and counit $\eta^{-1} : GF \to \mathrm{Id}_{\mathcal{C}}$.
The equality $\eta^{-1} G = G\epsilon$ implies the coherence condition $\gamma \circ \epsilon = \mathrm{Id}$.

Proposition

Let $\mathcal{C}$ be a category and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation. Then, the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection $\Leftrightarrow$ e splits, i.e. its components are all split idempotents (e.g. when $\mathcal{C}$ is idempotent complete, i.e. all idempotents split).

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($\Rightarrow$) H has a left and right adjoint equal, say $L : \mathcal{C}_e \to \mathcal{C}$, which is fully faithful, and such that the coherence condition $\eta \circ \epsilon' = \mathrm{Id}_{HL}$ is satisfied, where $\eta : \mathrm{Id} \to HL$ is the unit of $L \dashv H$, and $\epsilon' : HL \to \mathrm{Id}$ is the counit of $H \dashv L$. Denote by $\eta' : \mathrm{Id} \to LH$ and $\epsilon : LH \to \mathrm{Id}$ the unit and counit of $H \dashv L$ and $L \dashv H$, respectively. Then, $\eta' \circ \epsilon = \mathrm{Id}_{LH}$ and $e = \epsilon \circ \eta'$.

($\Leftarrow$) Check that H is a coreflection.

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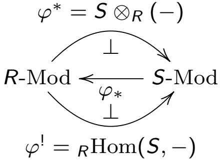
($\Leftarrow$) Check that H is a coreflection.

Corollary

A functor $F : \mathcal{C} \to \mathcal{D}$ factors as a bireflection followed by a separable functor $\Leftrightarrow$ it is semi-separable and the associated natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ splits. Moreover, any such a factorization is the same given by the coidentifier, up to a category equivalence.

Let $\varphi : R \to S$ be a morphism of rings and consider the *extension* φ^*, the *restriction* of scalars functor φ_*, and the *coinduction* functor $\varphi^!$:



Proposition

φ^* is semi-separable $\Leftrightarrow$ $\varphi^!$ is semi-separable $\Leftrightarrow$ $\exists\ \chi \in {}_R\mathrm{Hom}(S, R)_R$ such that $\varphi \circ \chi \circ \varphi = \varphi$, i.e. $\varphi\chi(1_S) = 1_S$.

Remark

$\varphi : R \to S$ is an epimorphism of rings $\Rightarrow$ φ^* is a reflection, and $\varphi^!$ is a coreflection. Thus, φ^* (and $\varphi^!$) is semi-separable $\Leftrightarrow$ it is naturally full $\Leftrightarrow$ it is Frobenius $\Leftrightarrow$ it is a bireflection.

Write $\varphi = \iota \circ \overline{\varphi}$, where $\iota : \varphi(R) \to S$ is the canonical inclusion and $\overline{\varphi} : R \to \varphi(R)$ is the corestriction of φ to its image $\varphi(R)$. Then,

$$\varphi^* = \iota^* \circ \overline{\varphi}^* \text{ is semi-separable} \Leftrightarrow \iota^* \text{ is separable and } \overline{\varphi}^* \text{ is a bireflection.}$$

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4. Eilenberg-Moore and Kleisli categories

Idempotent Completion

The idempotent completion of a category $\mathcal{C}$ is a category $\mathcal{C}^\natural$ whose objects are pairs (X, e), where X is an object in $\mathcal{C}$ and $e : X \to X$ is an idempotent in $\mathcal{C}$, and a morphism $f : (X, e) \to (X', e')$ in $\mathcal{C}^\natural$ is a morphism $f : X \to X'$ in $\mathcal{C}$ such that $f = e' \circ f \circ e$.

$\Rightarrow$ $\exists$ a canonical functor $\iota_\mathcal{C} : \mathcal{C} \to \mathcal{C}^\natural$, $X \mapsto (X, \mathrm{Id}_X)$, which is fully faithful; $\iota_\mathcal{C}$ is an equivalence if and only if $\mathcal{C}$ is idempotent complete.
Any functor $F : \mathcal{C} \to \mathcal{D}$ can be extended to a functor $F^\natural : \mathcal{C}^\natural \to \mathcal{D}^\natural$, by

$$F^\natural(X, e) = (FX, Fe),\ F^\natural f = Ff,$$

so that $\iota_\mathcal{D} \circ F = F^\natural \circ \iota_\mathcal{C}$.

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Proposition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then,

1) *F is faithful $\Leftrightarrow$ so is $F^\natural$.*
2) *F is full $\Leftrightarrow$ so is $F^\natural$.*
3) *F is semi-separable $\Leftrightarrow$ so is $F^\natural$.*

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$\Longrightarrow$

Corollary

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then,

1) F is separable $\Leftrightarrow$ so is $F^\natural$.
2) F is naturally full $\Leftrightarrow$ so is $F^\natural$.
3) F is fully faithful $\Leftrightarrow$ so is $F^\natural$.

Conditions up to retracts

Consider a functor $F : \mathcal{C} \to \mathcal{D}$ and its completion $F^\natural : \mathcal{C}^\natural \to \mathcal{D}^\natural$.

- F is an equivalence up to retracts if $F^\natural$ is an equivalence.

CHEN X.-W., A note on separable functors and monads with an application to equivariant derived categories. *Abh. Math. Semin. Univ. Hambg. 85*, 2015, no. 1, 43–52.

We say that

- F is a reflection up to retracts (resp. coreflection up to retracts) if $F^\natural$ is a reflection (resp. coreflection).
- F is a bireflection up to retracts if $F^\natural$ is a bireflection.

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Remark

1) F is an equivalence up to retracts $\Rightarrow$ it is both a reflection up to retracts, and a coreflection up to retracts. The same holds for any bireflection up to retracts.
2) F is a reflection (resp. coreflection) $\Rightarrow$ it is a reflection (resp. coreflection) up to retracts.
3) F is a semi-separable (co)reflection up to retracts $\Leftrightarrow$ F is a bireflection up to retracts.
4) F is a bireflection $\Rightarrow$ it is a bireflection up to retracts.
5) F is an equivalence up to retracts $\Leftrightarrow$ it is a fully faithful bireflection up to retracts.
6) F is an equivalence $\Rightarrow$ it is an equivalence up to retracts.

$\Rightarrow$ Let F be a (co)reflection up to retracts. Then, F is semi-separable $\Leftrightarrow$ it is naturally full.

Proposition

Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,

1. *G is a coreflection (resp. bireflection, equivalence) up to retracts $\Rightarrow$ it is a coreflection (resp. bireflection, equivalence).*
2. *F is a reflection (resp. bireflection, equivalence) up to retracts $\Rightarrow$ it is a reflection (resp. bireflection, equivalence).*

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Sketch Proof.

If G is a coreflection up to retracts, then it can be proved that there is $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta \circ \nu = \mathrm{Id}_{GF}$ and $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$, so that η is an isomorphism and G is a coreflection. Now, if G is a bireflection up to retracts, then G is a naturally full coreflection up to retracts, hence G is a naturally full coreflection, and this is also equivalent to say that G is a bireflection. Finally, an equivalence up to retracts is a fully faithful bireflection up to retracts, hence from the previous case it is a fully faithful bireflection, whence G is an equivalence. The results for F follow similarly.

Let $\mathcal{D}$ be an idempotent complete category and let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then, G has a left (resp. right) adjoint $\Leftrightarrow$ so has $G^\natural$.

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Corollary

Let $\mathcal{D}$ be an idempotent complete category and let $G : \mathcal{D} \to \mathcal{C}$ be a functor.

$\Rightarrow$ G is a coreflection (resp. reflection, bireflection, equivalence) up to retracts $\Leftrightarrow$ it is a coreflection (resp. reflection, bireflection, equivalence).

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Lemma

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors equipped with natural transformations $\eta : \mathrm{Id}_\mathcal{C} \to GF$ and $\epsilon : FG \to \mathrm{Id}_\mathcal{D}$.

1) *If there is a natural transformation $\nu : GF \to \mathrm{Id}_\mathcal{C}$ such that $\nu \circ \eta = \mathrm{Id}$ and $\nu G = G\epsilon \Rightarrow G$ is a coreflection up to retracts.*
2) *If there is a natural transformation $\gamma : \mathrm{Id}_\mathcal{D} \to FG$ such that $\epsilon \circ \gamma = \mathrm{Id}$ and $\gamma F = F\eta \Rightarrow F$ is a reflection up to retracts.*

Let $\mathcal{D}$ be an idempotent complete category and let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then, G has a left (resp. right) adjoint $\Leftrightarrow$ so has $G^\natural$.

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Let $\mathcal{D}$ be an idempotent complete category and let $G : \mathcal{D} \to \mathcal{C}$ be a functor.

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Example

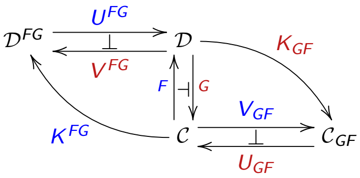
Let $\mathcal{C}$ be a category, let $e : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$ be an idempotent natural transformation. Then, the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection up to retracts.

Outline

1. Semi-separable functors
2. Semi-separability and (Co)reflections
3. Conditions up to retracts
4. Eilenberg-Moore and Kleisli categories

Eilenberg-Moore category

Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction. Consider the associated *Eilenberg-Moore* categories $\mathcal{C}_{GF}$ and $\mathcal{D}^{FG}$ with forgetful functors $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ and $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$ and



comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$,
$K_{GF} D = (GD, G\epsilon_D)$, $K_{GF} f = Gf$

cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$,
$K^{FG} C = (FC, F\eta_C)$, $K^{FG} f = Ff$

free functor $V_{GF} : \mathcal{C} \to \mathcal{C}_{GF}$,
$V_{GF} C = (GFC, G\epsilon_{FC})$, $V_{GF} f = GFf$

cofree functor $V^{FG} : \mathcal{D} \to \mathcal{D}^{FG}$,
$V^{FG} D = (FGD, F\eta_{GD})$, $V^{FG} f = FGf$

Then,

(i) G is semi-separable $\Leftrightarrow$ U_{GF} is separable (i.e., the monad $(GF, G\epsilon F, \eta)$ is separable) and K_{GF} is naturally full.

(ii) G is semi-separable (resp. separable, naturally full, fully faithful) $\Leftrightarrow$ so is V^{FG}.

(iii) F is semi-separable $\Leftrightarrow$ U^{FG} is separable (i.e., the comonad $(FG, F\eta G, \epsilon)$ is coseparable) and K^{FG} is naturally full.

(iv) F is semi-separable (resp. separable, naturally full, fully faithful) $\Leftrightarrow$ so is V_{GF}.

Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ, and let $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ be the comparison functor.

1. If the monad $(GF, G\epsilon F, \eta)$ is separable $\Rightarrow K_{GF}$ is a coreflection up to retracts.
2. If G is semi-separable $\Rightarrow K_{GF}$ is a bireflection up to retracts.
3. *(Chen X.-W., 2015)* If G is separable $\Rightarrow K_{GF}$ is an equivalence up to retracts.

Dually, let $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ be the cocomparison functor.

1. If the comonad $(FG, F\eta G, \epsilon)$ is coseparable $\Rightarrow K^{FG}$ is a reflection up to retracts.
2. If F is semi-separable $\Rightarrow K^{FG}$ is a bireflection up to retracts.
3. *(Chen X.-W., 2015)* If F is separable $\Rightarrow K^{FG}$ is an equivalence up to retracts.

(Co)comparison functors

Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ, and let $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ be the comparison functor.

1. If the monad $(GF, G\epsilon F, \eta)$ is separable $\Rightarrow K_{GF}$ is a coreflection up to retracts.
2. If G is semi-separable $\Rightarrow K_{GF}$ is a bireflection up to retracts.
3. *(Chen X.-W., 2015)* If G is separable $\Rightarrow K_{GF}$ is an equivalence up to retracts.

Dually, let $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ be the cocomparison functor.

1. If the comonad $(FG, F\eta G, \epsilon)$ is coseparable $\Rightarrow K^{FG}$ is a reflection up to retracts.
2. If F is semi-separable $\Rightarrow K^{FG}$ is a bireflection up to retracts.
3. *(Chen X.-W., 2015)* If F is separable $\Rightarrow K^{FG}$ is an equivalence up to retracts.

Corollary

1) Assume G is semi-separable. If K_{GF} has a left adjoint (it does when $\mathcal{D}$ is idempotent complete), then K_{GF} is a bireflection.
2) Assume F is semi-separable. If K^{FG} has a right adjoint (it does when $\mathcal{C}$ is idempotent complete), then K^{FG} is a bireflection.

Let $F : \mathcal{C} \to \mathcal{D}$ be a bireflection up to retracts. Consider the associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ and the corresponding factorization $F = F_e \circ H$.

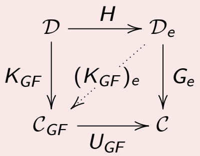
$\Rightarrow$ $F_e : \mathcal{C}_e \to \mathcal{D}$ is an equivalence up to retracts. If $\mathcal{C}$ is idempotent complete, then F_e is an equivalence.

Let $F : \mathcal{C} \to \mathcal{D}$ be a bireflection up to retracts. Consider the associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ and the corresponding factorization $F = F_e \circ H$.

$\Rightarrow$ $F_e : \mathcal{C}_e \to \mathcal{D}$ is an equivalence up to retracts. If $\mathcal{C}$ is idempotent complete, then F_e is an equivalence.

Corollary

Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction such that G is semi-separable. Then $\exists!$ functor $(K_{GF})_e : \mathcal{D}_e \to \mathcal{C}_{GF}$ such that $(K_{GF})_e \circ H = K_{GF}$ and $U_{GF} \circ (K_{GF})_e = G_e$.



$\Rightarrow$ $(K_{GF})_e$ is an equivalence up to retracts. If $\mathcal{D}$ is idempotent complete, then $(K_{GF})_e$ is an equivalence.

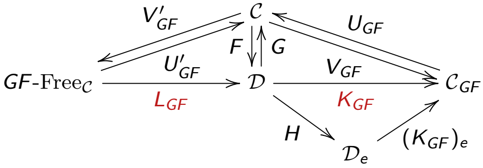
An analogus result holds for the cocomparison functor K^{FG} when F is semi-separable.

Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Consider the associated monad $(GF, G\epsilon F, \eta)$ and the *Kleisli category* $GF\text{-Free}_{\mathcal{C}}$ *of free GF-modules*:

- $\mathrm{Ob}(GF\text{-Free}_{\mathcal{C}}) = \mathrm{Ob}(\mathcal{C})$
- a morphism $f : C \to D$ in $GF\text{-Free}_{\mathcal{C}}$ is a morphism $f : C \to GF(D)$ in $\mathcal{C}$
- there is the *Kleisli comparison functor*

$$L_{GF} : GF\text{-Free}_{\mathcal{C}} \to \mathcal{D}, \quad C \mapsto F(C), \quad f \mapsto \epsilon_{FD} \circ F(f),$$

such that $K_{GF} \circ L_{GF} = J_{GF}$, where $J_{GF} : GF\text{-Free}_{\mathcal{C}} \to \mathcal{C}_{GF}$, $C \mapsto (GFC, G\epsilon_{FC})$, $f \mapsto G\epsilon_{FD} \circ GF(f)$ is fully faithful



Kleisli category

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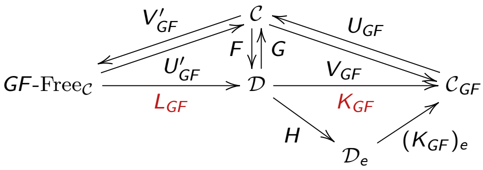
  such that $K_{GF} \circ L_{GF} = J_{GF}$, where $J_{GF} : GF\text{-}\mathrm{Free}_{\mathcal{C}} \to \mathcal{C}_{GF}$, $C \mapsto (GFC, G\epsilon_{FC})$, $f \mapsto G\epsilon_{FD} \circ GF(f)$ is fully faithful



- G is semi-separable $\Rightarrow K_{GF} \circ L_{GF}$ is an equivalence up to retracts. Moreover, also $H \circ L_{GF}$ is an equivalence up to retracts, so

$$GF\text{-}\mathrm{Free}_{\mathcal{C}}^{\natural} \cong \mathcal{D}_e^{\natural} \cong \mathcal{C}_{GF}^{\natural}$$

Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Consider the associated monad $(GF, G\epsilon F, \eta)$ and the *Kleisli category* $GF\text{-Free}_{\mathcal{C}}$ *of free GF-modules*:

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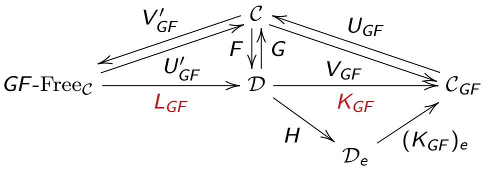

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$$GF\text{-Free}_{\mathcal{C}}^{\natural} \cong \mathcal{D}_e^{\natural} \cong \mathcal{C}_{GF}^{\natural}$$

- *(Balmer P., 2015)* G is separable $\Rightarrow$ L_{GF} and K_{GF} are equivalences up to retracts. Moreover, if $\mathcal{D}$ is idempotent complete, then K_{GF} is an equivalence, i.e. G is monadic.

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