Homotopy setoids and quotient completion

Cipriano Jr. Cioffo

UNIVERSITÀ DEGLI STUDI DI MILANO

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Outline

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1. Part 1
   - (Homotopy) Setoids
   - Elementary doctrines
   - Relative pretoposes
2. Part 2

Setoids in Dependent type theory

Definition

A *setoid* is a pair (X, R) where X is a *closed type* and R is a dependent type $x_1, x_2 : X \vdash R(x_1, x_2)$ which is an equivalence relation on X.

Intensional

$$\mathsf{Id}_A(a, b) \not\equiv a = b$$

- Decidability of type check.
- Strong normalization.
- N-canonicity.

Extensional

$$\mathsf{Id}_A(a, b) \equiv a = b$$

- Functional extensionality.
- UIP.
- Quotients.

Hofmann [4]: Ext.TT $\xhookrightarrow{SetoidModel}$ Int.TT

Homotopy setoids

h-level

0 $\text{is-contr}(C) := \sum_{c:C} \prod_{x:C} \text{Id}_A(c, x)$

1 $\text{is-prop}(P) := \prod_{x,y:P} \text{Id}_P(x, y)$

2 $\text{is-set}(A) := \prod_{x,y:A} \text{is-prop}(\text{Id}_A(x, y))$

n $\text{is-n+2-level}(X) := \prod_{x,y:X} \text{is-n+1-level}(\text{Id}_X(x, y))$

Definition

An *h-setoids* (X, R) is a setoid in which the type X is an h-set and the types $R(x_1, x_2)$ are h-propositions.

Categorical perspective

	Type	**Std**
OBJECTS	$\vdash X$	(X, R)
MAPS	$x : X \vdash f(x) : Y$	$\lfloor f \rfloor : (X, R) \to (Y, S)$

$$\begin{array}{ccc} \textbf{h-Sets} & \rightarrowtail & \textbf{h-Std} \\ \downarrow & & \downarrow \\ \textbf{Type} & \xrightarrow[X \to (X, \mathsf{Id}_X)]{} & \textbf{Std} \end{array}$$

Facts

- **Std** is the exact completion of the wlex category **Type**.
- **Std** is a Π-pretopos [7].

What about h-setoids?

Desiderata:

- (Local) cartesian closure
- Extensivity
- Well-behaved quotients of equivalence relations

Problem: Mismatch between "internal" and "external" notion of equivalence relation

$$r_1, r_2 : (X, R) \to (Y, S)$$

$$\rightsquigarrow$$

$$y_1, y_2 : Y \vdash \sum_{x:X} S(y_1, r_1(x)) \times S(y_2, r_2(x)).$$

Conseguence: **h-Std** is not Barr exact.

Possible solution: Change framework → Elementary doctrines!

Elementary doctrines

$$\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$$

- $\mathscr{C}$ has strict finite products
- For every $X \in \mathscr{C}$ there exists an element $\delta_X \in P(X \times X)$ with

$$\mathsf{P}(Y \times X) \underset{\perp}{\rightleftarrows} \mathsf{P}(Y \times X \times X)$$

Equivalently:

- $\top_X \leq \mathsf{P}_{\Delta_X}(\delta_X)$ $\qquad \vdash x = x$
- $\mathsf{P}(X) = \mathcal{D}es_{\delta_X}$ $\qquad x_1 = x_2, A(x_1) \vdash A(x_2)$
- $\delta_X \boxtimes \delta_Y \leq \delta_{X \times Y}$ $\qquad x_1 = x_2, y_1 = y_2 \vdash (x_1, y_1) = (x_2, y_2)$

Main Examples I

1. If $\mathscr{C}$ is lex then
$$\mathsf{Sub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{InfSL}$$

$$\mathsf{Sub}_{\mathscr{C}}(X) := \{\lfloor m \rfloor \mid m : M \rightarrowtail X\}$$
$$\mathsf{Sub}_{\mathscr{C}}(f) := f^*$$
$$\delta_X = \lfloor \Delta_X \rfloor$$

$$\begin{array}{ccc} P & \xrightarrow{f'} & M \\ {\scriptstyle f^*m}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ Y & \xrightarrow[f]{} & X \end{array}$$

2. If $\mathscr{C}$ is qlex (= strict finite products and weak pullbacks) then
$$\mathsf{PSub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{InfSL}$$

$$\mathsf{PSub}_{\mathscr{C}}(X) := (\mathscr{C}/X)_{po}$$
$$\mathsf{PSub}_{\mathscr{C}}(f) := f^*$$
$$\delta_X = \lfloor \Delta_X \rfloor$$

Main Examples II

3. F^{ML} : **Type**$^{op} \to$ InfSL

$$F^{ML}(X) := \{x : X \vdash B(x), \text{up to logical equivalence}\}$$

$$x : X \vdash B(x) \leftrightarrow B'(x) \ \textit{true}$$

if $y : Y \vdash t(y) : X$, then

$$F^{ML}(t)(B(x)) := B(t(y)).$$

$\delta_X = \mathsf{Id}_X$

4. $\mathcal{T}_{2,1}$: **h-Set**$^{op} \longrightarrow$ InfSL

$$\mathcal{T}_{2,1}(X) := \{x : X \vdash B(x) | \text{ is-prop(B) true}\}$$

$\delta_X = \mathsf{Id}_X$ is an h-proposition

Equivalence relations and quotients

Let P be an elementary doctrine:

- A P-eq. relation on $X \in \mathscr{C}$ is an element $\rho \in \mathsf{P}(X \times X)$ s.t.

$$\delta_X \leq \rho, \qquad \vdash_x x = x$$

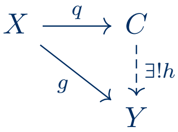
$$\mathsf{P}_{\langle 2,1\rangle}\rho \leq \rho, \qquad \rho(x_1, x_2) \vdash_{x_1,x_2} \rho(x_2, x_1)$$

$$\mathsf{P}_{\langle 1,2\rangle}\rho \wedge \mathsf{P}_{\langle 2,3\rangle}\rho \leq \mathsf{P}_{\langle 1,3\rangle}\rho, \qquad \rho(x_1, x_2), \rho(x_2, x_3) \vdash_{\bar{x}} \rho(x_1, x_3))$$

- A quotient of ρ is an arrow $q : X \to C$ s.t.

$$\rho \leq \mathsf{P}_{q \times q}\delta_C \qquad \rho(x_1, x_2) \vdash_{x_1,x_2} q(x_1) = q(x_2)$$

  and for all $g : X \to Y$ s.t $\rho \leq \mathsf{P}_{q \times q}\delta_Y$ there exists a unique arrow h



Elementary quotient completion

$$\overline{\mathsf{P}} : \overline{\mathscr{C}}^{op} \to \mathsf{InfSL}$$

	$\overline{\mathscr{C}}$	$\overline{\mathsf{P}}$
OBJECTS	(X, ρ)	$\overline{\mathsf{P}}(X,\rho) := \mathcal{D}es^{\star}_{\rho}$
MAPS	$\lfloor f \rceil : (X,\rho) \to (Y,\sigma)$	$\overline{\mathsf{P}}\lfloor f \rceil := \mathsf{P}_f$

$$^{\star}\mathcal{D}es_{\rho} = \{A(x) \in \mathsf{P}(X) | \rho(x_1, x_2), A(x_1) \vdash_{x_1,x_2} A(x_2)\}$$

Theorem (Maietti-Rosolini [6])

$$\textbf{\textit{ED}} \underset{U}{\overset{\overline{(-)}}{\rightleftarrows}} \textbf{\textit{QED}} \quad (\overline{(-)} \dashv U)$$

Main examples

1. If $\mathscr{C}$ is qlex then:

   $\mathsf{PSub}_{\mathscr{C}} : \mathscr{C}^{op} \to \mathsf{InfSL}$ $\quad$ $\overline{\mathsf{PSub}_{\mathscr{C}}} \cong \mathsf{Sub}_{\mathscr{C}_{ex}} : \mathscr{C}_{ex}^{op} \to \mathsf{InfSL}$

   Pseudo eq. relations

   $$R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X \quad \longleftrightarrow \quad \text{P-eq. relations } \lfloor < r_1, r_2 >: R \to X \times X \rfloor$$

2. $F^{ML} : \mathbf{Type}^{op} \to \mathsf{InfSL}$ $\quad$ $\overline{F^{ML}} : \mathbf{Std}^{op} \to \mathsf{InfSL}$

3. $\mathcal{T}_{2,1} : \mathbf{h\text{-}Set}^{op} \longrightarrow \mathsf{InfSL}$ $\quad$ $\overline{\mathcal{T}_{2,1}} : \mathbf{h\text{-}Std}^{op} \to \mathsf{InfSL}$

Review of theorems about LCC and Extensivity

Theorem (Carboni-Rosolini [1],Emmenegger [2])

If $\mathscr{C}$ is a wlex and has right adjoint to weak pullback functors, then TFAE:
i) Every slice $\mathscr{C}/A$ has extentional exponentials,
ii) $\mathscr{C}_{ex}$ is locally cartesian closed.

Theorem (Gran-Vitale, [3])

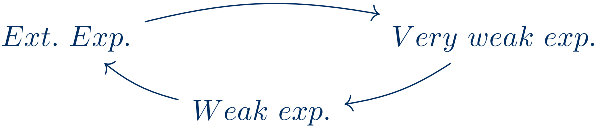
If $\mathscr{C}$ is wlex with sums, then TFAE:
i) $\mathscr{C}$ is weakly lextensive,
ii) $\mathscr{C}_{ex}$ is extensive.

LCC

Theorem

If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *is an elementary doctrine with* $\exists, \forall, \implies$ *and weak full comprehensions and comprehensive diagonals, then TFAE:*
i) Every slice $\mathscr{C}/A$ *has extentional exponentials,*
ii) $\overline{\mathscr{C}}$ *is locally cartesian closed.*

- Maietti-Pasquali-Rosolini, [5]. Slice-wise weakly cartesian closed.



Extensivity

Theorem

If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Frm}$ *is an elementary doctrine with* $\exists$ *and full weak comprehensions and comprehensive diagonals, then TFAE:*
i) $\mathscr{C}$ *is weakly lextensive,*
ii) $\overline{\mathscr{C}}$ *is extensive.*

- $A + B$ vs. $A \vee B$:
  - Assume + in the contexts $\implies$ $\mathscr{C}$ with coproducts.
  - Assume $\vee$ in the logic $\implies$ $\mathsf{P}(X) \in \mathsf{Frm}$.
  - Define a notion of weakly lextensive w.r.t. a doctrine.

Relative pretoposes

Definition

A *relative pretopos* is an extensive category $\mathscr{C}$ equipped with an elementary doctrine P in **QED**.

Example: Every pretopos $\mathscr{C}$ is relative to $\mathrm{Sub}_{\mathscr{C}}$.

Theorem

h-Std *is a* Π*-pretopos relative to* $\overline{\mathcal{T}_{2,1}}$.

Current investigations:

- **h-Std** as models of suitable type theories: $\mathbf{TT}_{IQ}$, $\mathbf{TT}_{EQ}$, **qmTT**
- Internal logic of **h-Std**

Outline

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1. Part 1

2. Part 2
   - Weak doctrines
   - Proof-irrelevant elements

Towards a generalization

Desiderata:

- Generalization of the theorems about lcc and extensivity
- A direct proof of the lcc
- A framework to which contains the "slice" of a doctrine P/A
- An internal description of *pseudo equivalence relations* for wlex categories

Problems: Weak finite products!

Weak finite products

Definition

A weak product of $X, Y \in \mathscr{C}$ is an object $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\rightarrow} Y$ such that

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{f} & \Big\downarrow{\scriptstyle \exists\not!} & \searrow^{g} & \\ X & \underset{\mathsf{p}_1}{\longleftarrow} & W & \underset{\mathsf{p}_2}{\longrightarrow} & Y \end{array}$$

Examples:

- In **Set**, given two sets X, Y then $X \times Y \times \{0, 1\}$ is a weak product of X and Y.
- In **Type**/A

$$\begin{array}{ccc} \sum_{x:X, y:Y} \mathsf{Id}_A(f(x), g(y)) & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ X & \underset{f}{\longrightarrow} & A. \end{array}$$

Weak elementary doctrines

$$\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$$

- $\mathscr{C}$ has weak finite products
- For every $X \in \mathscr{C}$ and weak product $X \overset{\mathsf{p}_1}{\leftarrow} W \overset{\mathsf{p}_2}{\to} X$ there exists an element $\delta_X^W \in \mathsf{P}(W)$ s.t.

  - $\top_Z \leq \mathsf{P}_d(\delta_X^W)$
  - $\mathsf{P}(X) = \mathcal{D}es_{\delta_X^W}$
  - $\delta_Y^V \leq \mathsf{P}_{f\times f}\delta_X^W$
  - $\delta_X^W \in \mathcal{D}es_{\delta_X^W \boxtimes \delta_X^W}$

  - $\top_X \leq \mathsf{P}_{\Delta_X}(\delta_X)$
  - $\mathsf{P}(X) = \mathcal{D}es_{\delta_X}$
  - $\delta_X \boxtimes \delta_Y \leq \delta_{X\times Y}$

Examples I

1. Every elementary doctrine P is a weak elementary doctrine. If $X \overset{p_1}{\leftarrow} W \overset{p_2}{\rightarrow} X$ is a weak product then there exists a unique arrow $< p_1, p_2 >: W \to X \times X$

$$\delta_X^W := P_{<p_1,p_2>}\delta_X$$

2. If $\mathscr{C}$ is wlex then the functor $PSub_{\mathscr{C}} : \mathscr{C}^{op} \to InfSL$ is a weak elementary doctrine and

$$\delta_X^W := \lfloor e \rceil$$

where

$$E \xrightarrow{e} W \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$$

is a weak equalizer of p_1, p_2.

Examples II

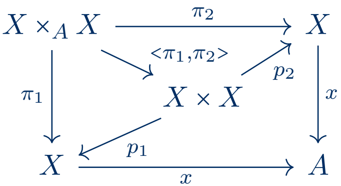
3. If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ is a (weak) elementary doctrine with weak comprehensions and comprehensive diagonals and $A \in \mathscr{C}$ then the *slice doctrine* is a weak elementary doctrine:

$$\mathsf{P}_{/A} : \mathscr{C}/A^{op} \to \mathsf{InfSL}$$

$$\mathsf{P}_{/A}(x : X \to A) := \mathsf{P}(X)$$
$$\mathsf{P}_{/A}(f : y \to x) := \mathsf{P}_f$$

$\mathsf{P}_{/A}(w) = \mathsf{P}(X \times_A X)$ and

$$\delta^w_x := \mathsf{P}_{<\pi_1,\pi_2>}\delta_X$$



Key differences with "strict" elementary doctrines

- Two weak products W, W' of the same elements $X, Y \in \mathscr{C}$ are not necessarily isomorphic.
- The fibers $\mathsf{P}(W)$ and $\mathsf{P}(W')$ are not necessarily isomorphic.
- Given two arrows $f : Z \to X$ and $g : Z \to Y$ the weak u.p. implies the existence of a not necessarily unique arrow $< f, g >: Z \to W$. The reindexings $\mathsf{P}_{<f,g>}$ and $\mathsf{P}_{<f,g>'}$ are not necessarily equal.
- We have only the inequality

$$\delta_{X\times Y} \leq \delta_X \boxtimes \delta_Y$$

Intuition: $x_1 = x_2, y_1 = y_2 \not\Longrightarrow ((x_1, y_1), p) = ((x_2, y_2), q)$
$\delta_{X\times Y} \sim$ *proof-relevant* equality
$\delta_X \boxtimes \delta_Y \sim$ *proof-irrelevant* or *component-wise* equality

Proof-irrelevant elements

Definition

If W is a weak product of $X, Y \in \mathscr{C}$ the *proof-irrelevant* elements of W are the sub-poset of $\mathsf{P}(W)$ given by $\mathsf{PIrr}(W) := \mathcal{D}es_{\delta_X \boxtimes \delta_Y}$

- Different weak products have isomorphic proof-irrelevant elements: If W, W' are weak products of $X, Y \in \mathscr{C}$ then there exists an arrow $W' \xrightarrow{h} W$ s.t. $\mathsf{p}_i \circ h = \mathsf{p'}_i$

$$\begin{array}{ccc} \mathsf{PIrr}(W) & \xrightarrow{\cong} & \mathsf{PIrr}(W') \\ \downarrow & & \downarrow \\ \mathsf{P}(W) & \xrightarrow[\mathsf{P}_h]{} & \mathsf{P}(W') \end{array}$$

- Proof-irrelevant elements are *reindexed by projections*.
- Up to iso: we denote proof-irrelevant elements of X and Y with $\mathsf{P}^s[X, Y]$

Motivational example

- In $F^{ML}_{/A} : \mathbf{Type}/A^{op} \to \mathsf{InfSL}$, if

$$\begin{array}{ccc} W := \sum_{x:X, y:Y} \mathsf{Id}_A(f(x), g(y)) & \xrightarrow{\pi_2} & Y \\ \pi_1 \downarrow & & \downarrow g \\ X & \xrightarrow[f]{} & A \end{array}$$

$$F^{ML}_{/A}\mathsf{Irr}(W) = \{(x, y, p) : W \vdash R(x, y, p) |$$

$$\mathsf{Id}_X(x, x'),\ \mathsf{Id}_Y(y, y'),\ P(x, y, p)\ \vdash P(x', y', p')\}.$$

Up to iso $\sim F^{ML}(X \times Y)$.

Strictification and quotient completion

If $\mathscr{C}$ is a category, we can freely add strict finite products and obtain the category $\mathscr{C}_s$:

Obj. Finite lists $[X_i]_{i\in[n]}$

Arr. $(f,\hat{f}) : [X_i]_{i\in[n]} \to [Y_j]_{j\in[m]}$

$$\mathbf{WED} \underset{U}{\overset{(-)^s}{\rightleftarrows}} \mathbf{ED} \qquad (\bot)$$

A P-eq. relation over $X \in \mathscr{C}$ is an element $\rho \in \mathsf{P}^s[X,X]$ satisfying ref., sym. and tra.

$$\mathbf{WED} \xrightarrow[\mathsf{P}\to\overline{\mathsf{P}}]{} \mathbf{QED}$$

Not left bi-adjoint to the forgetful 2-functor!

$$\begin{array}{ccc} \mathbf{WED} & \xrightarrow{(-)^s} & \mathbf{ED} \\ & \not\circlearrowleft & \\ {\scriptstyle\overline{(-)}}\searrow & & \swarrow{\scriptstyle\overline{(-)}} \\ & \mathbf{QED} & \end{array}$$

Main Theorems generalization

Theorem

If $\mathscr{C}$ *is wlex then* $\overline{\mathsf{PSub}_{\mathscr{C}}} \cong \mathsf{Sub}_{\mathscr{C}_{ex}} : \mathscr{C}_{ex}^{op} \to \mathsf{InfSL}$

Theorem

If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{InfSL}$ *is a weak elementary doctrine with* $\exists, \forall, \implies$ *and weak full comprehensions and comprehensive diagonals, then TFAE:*

i) Every slice $\mathscr{C}/A$ *has extentional exponentials,*

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Theorem

If $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Frm}$ *is a weak elementary doctrine with* $\exists$ *and full weak comprehensions and comprehensive diagonals, then TFAE:*

i) $\mathscr{C}$ *is weakly lextensive,*

ii) $\overline{\mathscr{C}}$ *is extensive.*

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