Semi-abelian categories, Hopf algebras and internal groupoids

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Outline

"Abelian" versus "semi-abelian"

Internal groupoids in algebra

Torsion theories, groupoids and exact completion

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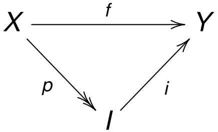
Definition
A category $\mathbb{C}$ is abelian if

- $\mathbb{C}$ has a zero-object 0
- $\mathbb{C}$ has binary products $A \times B$
- any arrow f in $\mathbb{C}$ has a factorisation $f = i \circ p$



where p is a *normal epi* and i is a *normal mono*.

Normal monomorphism

An arrow $k\colon K \to A$ is called a normal monomorphism if it is the kernel of some arrow in $\mathbb{C}$: there is an $f\colon A \to B$ such that

$$\begin{array}{ccc} K & \xrightarrow{k} & A \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ 0 & \longrightarrow & B \end{array}$$

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In the category Grp of groups :

- normal monomorphism = normal subgroup

In the category Ab of abelian groups :

- any monomorphism $k\colon K \to A$ is normal !

Dually :

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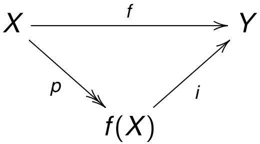
Examples

In the categories Grp and Ab :

- normal epimorphism = surjective homomorphism.

The category Ab of abelian groups is abelian :

- Ab has a 0-object : the trivial group $\{0\}$
- the product $A \times B$ exists for any A, B
- any homomorphism f in Ab has a factorisation $f = i \circ p$



where p is a *surjective homomorphism* and i is an inclusion as a *normal subgroup*.

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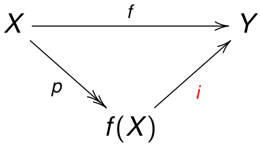
Grp is not abelian :

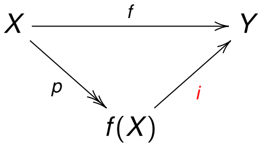
- it has a 0-object : the trivial group $\{1\}$
- the product $A \times B$ exists for any $A, B \in$ Grp
- Problem : an arrow f in Grp does not have a factorisation $f = i \circ p$



with p a *surjective homomorphism* and i an inclusion as a **normal subgroup**.

The category Rng of rings is not abelian :

- an arrow f in Rng does not have a factorisation $f = i \circ p$



with p a *surjective homomorphism* and i an inclusion as an **ideal**.

Question : is there a list of simple axioms to develop a unified treatment of the categories Grp, Rng, Lie_K ?

S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. (1950)

The problem was to find the "fourth proportional" in

$$\text{Ab} : \text{abelian category} \quad = \quad \text{Grp} : ?$$

The problem was to find the “fourth proportional” in

Ab : abelian category = Grp : ?

Aim : find an axiomatic context for

- Noether’s isomorphism theorems
- non-abelian homological algebra
- radical theory
- commutator theory

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A finitely complete category $\mathbb{C}$ is semi-abelian if

- $\mathbb{C}$ has a 0 object
- $\mathbb{C}$ has $A + B$
- $\mathbb{C}$ is (Barr) exact
- $\mathbb{C}$ is (Bourn) protomodular :

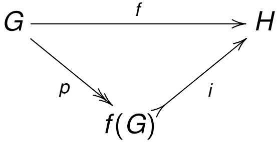
$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{k} & A & \underset{f}{\overset{\longleftarrow}{\longrightarrow}} & B \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ 0 & \longrightarrow & K' & \xrightarrow[k']{} & A' & \underset{f'}{\overset{\longleftarrow}{\longrightarrow}} & B' \end{array}$$

u, w isomorphisms $\Rightarrow$ v isomorphism.

Example

The category Grp is semi-abelian :

- every homomorphism f in Grp has a factorisation $f = i \circ p$



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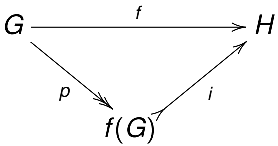

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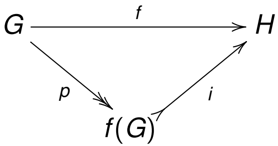

  where p is a regular epimorphism and i is a monomorphism ;
- these factorisations are pullback stable ;
- Grp is exact ;
- the Split Short Five Lemma holds in Grp.

Examples

Grp, Rng, Alg_K, Lie_K (more generally, any variety of Ω-groups) are all semi-abelian categories.

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Any abelian category !

Terminology :

[$\mathbb{C}$ is abelian] $\Leftrightarrow$ [$\mathbb{C}$ and $\mathbb{C}^{op}$ are semi-abelian] !

Crossed modules

A crossed module is a group homomorphism $A \xrightarrow{\alpha} B$ with an action of B on A such that :

- $\alpha({}^{b}a) = b\alpha(a)b^{-1}$, for all $a \in A$, $b \in B$
- ${}^{\alpha(a)}a_1 = aa_1a^{-1}$, for all $a, a_1 \in A$.

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A morphism of crossed modules is a pair (f_1, f_0) of group homomorphisms making the square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ A' & \xrightarrow[\alpha']{} & B' \end{array}$$

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The category XMod of crossed modules is semi-abelian :

$$\mathsf{XMod} \cong \mathsf{Grpd(Grp)}$$

Cocommutative Hopf algebras

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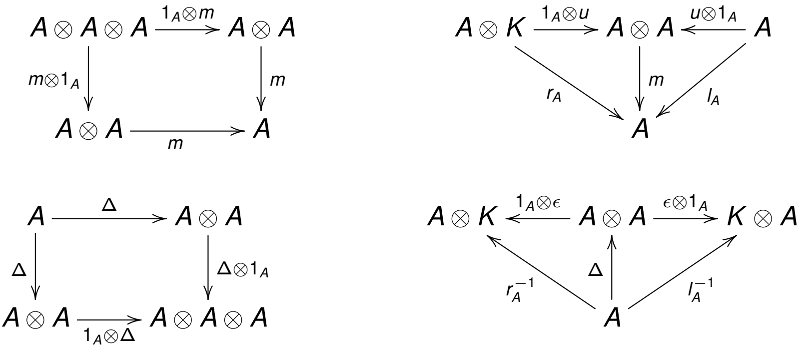
Bialgebras

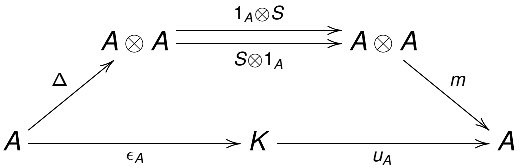
A K-bialgebra $(A, m, u, \Delta, \epsilon)$ is both a K-algebra (A, m, u) and a K-coalgebra (A, Δ, ϵ), where m, u, Δ, ϵ are linear maps such that



commute, and m and u are K-coalgebra morphisms.

Cocommutative Hopf algebras

A Hopf K-algebra $(A, m, u, \Delta, \epsilon, S)$ is a K-bialgebra with a linear map $S\colon A \to A$, the antipode, making the following diagram commute :



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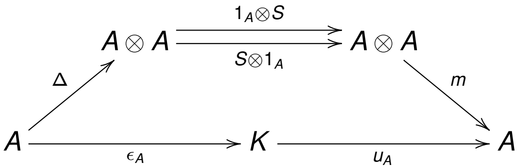
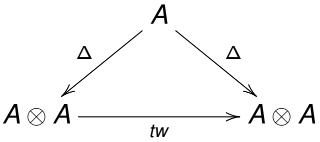


$(A, m, u, \Delta, \epsilon, S)$ is cocommutative if the following triangle commutes :



In Sweedler's notations : $\Delta(a) = a_1 \otimes a_2 = a_2 \otimes a_1$, for any $a \in A$.

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$A \in \mathsf{Hopf}_{K,coc}$ is abelian $\Leftrightarrow \Delta \colon A \rightarrow A \otimes A$ is a normal mono

$\Leftrightarrow$ A is commutative : $ab = ba$

$\Leftrightarrow$ $A \in \mathsf{Hopf}_{K,coc}^{comm}$ $\square$

In full generality, when $\mathbb{C}$ is a semi-abelian category, $\mathsf{Ab}(\mathbb{C})$ is an abelian category !

$$\mathsf{Ab}(\mathbb{C}) \underset{U}{\overset{ab}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbb{C}$$

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$\mathsf{Ab}(\mathbb{C})$ is a full reflective subcategory of $\mathbb{C}$, where the left adjoint $\mathsf{ab}\colon \mathbb{C} \to \mathsf{Ab}(\mathbb{C})$ is an “abelianisation functor” :

$$\mathsf{ab}(A) = \frac{A}{[A, A]},$$

where $[A, A]$ is a suitably defined normal subobject of A.

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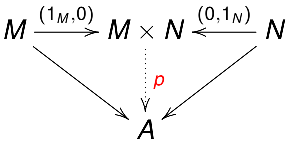
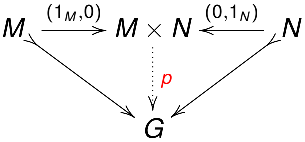
Definition (Huq, 1968)

$[M, N] = 0$ if and only if there is a morphism $p\colon M \times N \to A$ such that the following diagram commutes :



The case of groups

If $M \to G$ and $N \to G$ are (the inclusions of) two normal subgroups of a group G, the existence of a group morphism p making the diagram



commute implies that $[M, N] = \{1\}$:

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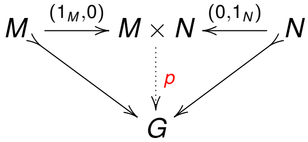

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Conversely, the condition $[M, N] = \{1\}$ implies that the map

$$p(m, n) = m \cdot n, \qquad \forall m \in M, n \in N$$

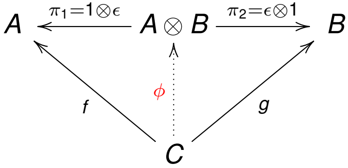
is a group morphism.

The case of cocommutative Hopf algebras

In the category $\text{Hopf}_{K,coc}$ of cocommutative Hopf algebras the categorical product is the tensor product : given two Hopf algebra morphisms $f\colon C \to A$ and $g\colon C \to B$ there is a unique morphism

$$\phi = (f \otimes g) \circ \Delta$$

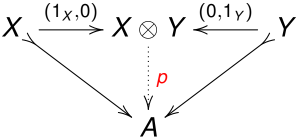
making the following diagram commute :



Lemma

Given two Hopf subalgebras $X \to A$ and $Y \to A$ in $\mathsf{Hopf}_{K,coc}$ the following conditions are equivalent :

- there is a morphism $p \colon X \otimes Y \to A$ such that the following diagram commutes :



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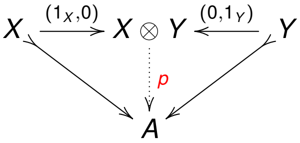

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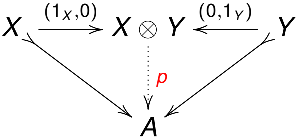


- $xy = yx$, for any $x \in X, y \in Y$,
- $x_1 y_1 S(x_2) S(y_2) = \epsilon(x)\epsilon(y)$, for any $x \in X, y \in Y$.

The Hopf subalgebra

$$[X, Y] = \langle\{x_1 y_1 S(x_2) S(y_2) \mid x \in X, y \in Y\}\rangle_A$$

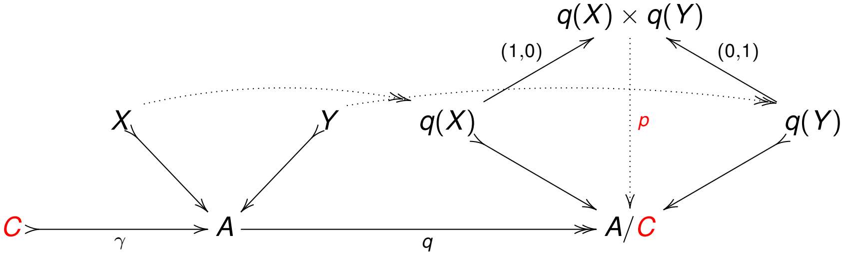
of A generated by all commutators

$$x_1 y_1 S(x_2) S(y_2)$$

satisfies the universal property defining the categorical commutator.

In a semi-abelian $\mathbb{C}$ the commutator $[X, Y]$ of X and Y in A is the smallest normal subobject $\gamma\colon C \to A$ of A such that the images $q(X)$ and $q(Y)$ along the quotient $q\colon A \to \frac{A}{C}$ "commute" :

$$[q(X), q(Y)] = 0.$$



Several classical results hold true in a semi-abelian category :

- the Noether isomorphism theorems
- the Snake Lemma and the 3×3-Lemma (D. Bourn, 2001)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K_1 & \rightarrow & A_1 & \rightarrow & B_1 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K_2 & \rightarrow & A_2 & \rightarrow & B_2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K_3 & \rightarrow & A_3 & \rightarrow & B_3 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

- the Jordan-Hölder theorem (F. Borceux and M. Grandis, 2007)

The “general idea” :

Whereas

abelian = exact + *additive*

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abelian = exact + *additive*

the “non-additive” version of this “equation” is

semi-abelian = exact + 0 + binary coproducts + *protomodular*

Outline

“Abelian” versus “semi-abelian”

Internal groupoids in algebra

Torsion theories, groupoids and exact completion

The abelian case

In an abelian category $\mathbb{C}$ any reflexive graph

$$X_1 \underset{c}{\overset{d}{\rightrightarrows}} X_0 \quad (e : X_0 \to X_1) \qquad d \cdot e = 1_{X_0} = c \cdot e$$

is naturally equipped with an internal groupoid structure :

$$X_1 \times_{X_0} X_1 \underset{p_2}{\overset{p_1}{\rightrightarrows}} X_1 \underset{c}{\overset{d}{\rightrightarrows}} X_0 \; . \qquad (m : X_1 \times_{X_0} X_1 \to X_1,\; s : X_1 \to X_1,\; e : X_0 \to X_1)$$

The forgetful functor $\mathsf{Grpd}(\mathbb{C}) \to \mathsf{RG}(\mathbb{C})$ is a category isomorphism : the “Lawvere condition”.

For instance, in Ab, the (object part) of the pullback is given by

$$
\begin{aligned}
X_1 \times_{X_0} X_1 &= \{(f,g) \in X_1 \times X_1 \mid c(f) = d(g)\} \\
&= \{\, (X \xrightarrow{f} Y,\ Y \xrightarrow{g} Z)\, \}.
\end{aligned}
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In the category Ab of abelian groups the multiplication

$$m\colon X_1 \times_{X_0} X_1 \to X_1$$

is (uniquely) defined by

$$m(f,g) = g - 1_Y + f.$$

This is not the case in the category Grp of groups : a reflexive graph

$$X_1 \underset{c}{\overset{d}{\rightrightarrows}} X_0 \quad (e : X_0 \to X_1)$$

in Grp is a groupoid if and only if $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = \{1\}$.

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Grpd(Grp) is a reflective subcategory of RG(Grp)

$$\mathrm{Grpd(Grp)} \underset{U}{\overset{F}{\underset{\perp}{\leftrightarrows}}} \mathrm{RG(Grp)}.$$

For any reflexive graph X, the reflector F is defined by a quotient :

$$\begin{array}{ccc} X_1 & \xrightarrow{\eta_{X_1}} & \dfrac{X_1}{[\mathrm{Ker}(d),\mathrm{Ker}(c)]} \\ d \downarrow \uparrow e \downarrow c & & \bar{d} \downarrow \uparrow \bar{e} \downarrow \bar{c} \\ X_0 & \underset{1_{X_0}}{=\!=\!=} & X_0 \end{array}$$

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Lemma

A reflexive graph in $\mathsf{Hopf}_{K,coc}$

$$X_1 \underset{c}{\overset{d}{\rightrightarrows}} X_0 \quad (e\colon X_0 \to X_1) \qquad d \cdot e = 1_{X_0} = c \cdot e$$

has a groupoid structure if and only if

$$[\mathrm{Ker(d)}, \mathrm{Ker(c)}] = 0.$$

Equivalently, one has that $xy = yx$ for any $x \in \mathrm{Ker}(d)$ and $y \in \mathrm{Ker}(c)$, where

$$\mathrm{Ker}(d) = \{x \in X_1 \mid x_1 \otimes d(x_2) = x \otimes 1\}.$$

When B is cocommutative, a B-module Hopf algebra X is a (cocommutative) Hopf algebra X equipped with a linear map $\xi\colon B \otimes X \to X$, with $\xi(b \otimes x) = {}^{b}x$, such that

- ${}^{(bb')}x = {}^{b}({}^{b'}x)$
- ${}^{1_B}x = x$
- ${}^{b}xy = {}^{b_1}x\,{}^{b_2}y$
- ${}^{b}1_X = \epsilon(b)1_X$
- $({}^{b}x)_1 \otimes ({}^{b}x)_2 = {}^{b_1}x_1 \otimes {}^{b_2}x_2$
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Definition (S. Majid 2012)

A crossed module of (cocommutative) Hopf algebras is a morphism

$$X \xrightarrow{\ d\ } B,$$

where $B \in \mathsf{Hopf}_{K,coc}$, X is a B-module Hopf algebra such that

$$d({}^{b}x) = b_1 d(x) S(b_2), \qquad {}^{d(y)}x = y_1 x S(y_2), \qquad \forall x, y \in X, \forall b \in B.$$

There is a category of crossed modules of (cocommutative) Hopf algebras, denoted by HXMod.

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Lemma (M. Gran, F. Sterck, J. Vercruysse 2019)

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The categories HXMod *and* $\mathrm{Grpd}(\mathrm{Hopf}_{K,coc})$ *are equivalent.*

The proof of this result uses the "normalization functor" $N\colon \mathrm{Grpd}(\mathrm{Hopf}_{K,coc}) \to \mathrm{HXMod}$ sending a groupoid

$$X_1 \underset{c}{\overset{d}{\rightrightarrows}} B \quad (e\colon B \to X_1)$$

to the Hopf algebra morphism

$$\mathrm{Ker(d)} \xrightarrow{\mathrm{Ker}\,(d)} X_1 \xrightarrow{c} B\ ,$$

where the action $B \otimes \mathrm{Ker}(d) \to \mathrm{Ker}(d)$ is defined by

$${}^{b}k = e(b_1) \cdot k \cdot e(S(b_2)), \qquad \forall b \in B, \forall k \in \mathrm{Ker}(d).$$

Theorem (D. Bourn, M. Gran 2002)

$\mathbb{C}$ is semi-abelian if and only if the category $\mathrm{Grpd}(\mathbb{C})$ *of internal groupoids in $\mathbb{C}$ is semi-abelian.*

The category HXMod is then semi-abelian.

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One can iterate the construction to define the category of double groupoids :

$$\mathrm{Grpd}(\mathrm{Grpd}(\mathrm{Hopf}_{K,coc})) \cong \mathrm{Grpd}^2(\mathrm{Hopf}_{K,coc}),$$

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It is possible to describe double groupoids in $\mathsf{Hopf}_{K,coc}$ in terms of crossed squares of Hopf algebras (F. Sterck, 2021) :

$$\mathsf{Grpd}^2(\mathsf{Hopf}_{K,coc}) \cong \mathsf{XMod}^2(\mathsf{Hopf}_{K,coc})$$

Outline

“Abelian” versus “semi-abelian”

Internal groupoids in algebra

Torsion theories, groupoids and exact completion

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1. $\mathcal{T}$ and $\mathcal{F}$ are full (replete) subcategories of $\mathbb{C}$;
2. for every object $C \in \mathbb{C}$ there is a short exact sequence

$$0 \longrightarrow T(C) \longrightarrow C \longrightarrow F(C) \longrightarrow 0$$

with $T(C) \in \mathcal{T}$ and $F(C) \in \mathcal{F}$;

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   with $T(C) \in \mathcal{T}$ and $F(C) \in \mathcal{F}$;
3. if $T \in \mathcal{T}$ and $X \in \mathcal{F}$ then the only morphism from T to X is

$$T \to 0 \to X.$$

Remark that the canonical short exact sequence

$$0 \longrightarrow T(C) \longrightarrow C \xrightarrow{\eta_C} F(C) = \frac{C}{T(C)} \longrightarrow 0$$

gives the reflection $F\colon \mathbb{C} \to \mathcal{F}$, and $\mathcal{F}$ is then normal epi-reflective :

$$\mathcal{F} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{C} \quad (F \dashv U)$$

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Example

$$\mathsf{Ab}_{t.f.} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{Ab}$$

where Ab is the category of abelian groups, $\mathsf{Ab}_{t.f.}$ is the category of torsion-free abelian groups.

Torsion-free subcategories of an abelian category $\mathbb{C}$ correspond to (normal epi)-reflective semi-localizations $\mathcal{F}$ of $\mathbb{C}$

$$\mathcal{F} \underset{U}{\overset{F}{\leftrightarrows}} \mathbb{C}. \quad (F \dashv U)$$

This means that $F \colon \mathbb{C} \to \mathcal{F}$ is semi-left-exact (Cassidy-Hébert-Kelly, 1985) : it preserves all pullbacks of the form

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & U(X) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle U(x)} \\ C & \xrightarrow[\eta_C]{} & UF(C) \end{array}$$

where $x \colon X \to F(C)$ lies in $\mathcal{F}$.

Theorem (W. Rump, 2001)

For a category $\mathcal{F}$ the following conditions are equivalent :

1. $\mathcal{F}$ is a torsion-free subcategory of an abelian category $\mathbb{C}$

$$\mathcal{F} \;\underset{U}{\overset{F}{\underset{\perp}{\leftrightarrows}}}\; \mathbb{C};$$

2. (a) $\mathcal{F}$ is additive ;
   (b) any morphism $f\colon A \to D$ in $\mathcal{F}$ has a factorization $f = kgq$

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & D \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle k} \\ B & \xrightarrow[\;g\;]{} & C \end{array}$$

   with q a normal epi, g a bimorphism, k a normal mono ;
   (c) normal epimorphisms are pullback stable.

A category $\mathcal{F}$ satisfying the conditions (a), (b) and (c) is called an almost abelian category.

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Examples

Any abelian category, Ab(Top), Ab(Haus), Banach spaces, locally compact abelian groups, Mono(Ab), etc.

New examples of torsion theories have been studied in the categories $\mathsf{Grp}, \mathsf{Grp}(\mathsf{Comp}), \mathsf{CRng}, \mathsf{Grpd}(\mathsf{Grp})$, $\mathsf{Hopf}_{K,coc}$.

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Question

Can we find a similar characterization in the semi-abelian context ?

A crucial result is the following :

Lemma

Let $\mathcal{F}$ be a regular category. Then :
$\mathcal{F}$ is protomodular $\Leftrightarrow$ its exact completion $\mathcal{F}_{\text{ex/reg}}$ is protomodular.

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Let $\mathcal{F}$ be a regular category. Then :
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Theorem (M. Gran and S. Lack, 2016)

- **(a)** *$\mathcal{F}$ is a semi-localization of an exact protomodular category $\mathbb{C}$;*
- **(b)** *$\mathcal{F}$ is regular, is a semi-localization of its exact completion $\mathcal{F}_{\text{ex/reg}}$, and $\mathcal{F}_{\text{ex/reg}}$ is protomodular ;*
- **(c)** *$\mathcal{F}$ is regular, protomodular, and has stable coequalizers of equivalence relations.*

To prove this we also use the following crucial result :

Theorem (S. Mantovani, 1998)

For a category $\mathcal{F}$ the following conditions are equivalent :

1. *$\mathcal{F}$ is a semi-localization of an exact category $\mathbb{C}$;*
2. *$\mathcal{F}$ has finite limits and stable coequalizers of equivalence relations.*

Given a coequalizer $q\colon A \to B$ of an equivalence relation

$$R \underset{p_2}{\overset{p_1}{\rightrightarrows}} A \overset{q}{\twoheadrightarrow} B$$

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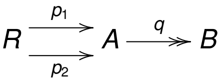

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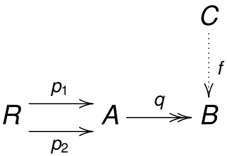

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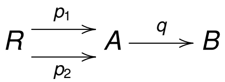

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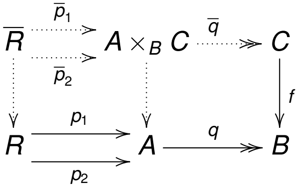

Given in $\mathcal{X}$ a coequalizer $q\colon A \to B$ of an equivalence relation

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and any arrow f

$$\begin{array}{ccccc} \overline{R} & \underset{\overline{p}_2}{\overset{\overline{p}_1}{\rightrightarrows}} & A \times_B C & \overset{\overline{q}}{\twoheadrightarrow} & C \\ \downarrow & & \downarrow & & \downarrow f \\ R & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & A & \overset{q}{\twoheadrightarrow} & B \end{array}$$

$\mathcal{F}$ has stable coequalizers $\quad \Leftrightarrow \quad \overline{q} = \mathrm{coeq}(\overline{p}_1, \overline{p}_2)$

Example

The category RedCRng of reduced rings ($x^n = 0 \Rightarrow x = 0$) is a semi-localization of a semi-abelian category.

In this case :

$$\text{RedCRng}_{\text{ex/reg}} = \text{CRng}$$

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In this case :

$$\mathsf{RedCRng}_{\mathsf{ex/reg}} = \mathsf{CRng}$$

Example

Any torsion-free subcategory $\mathcal{F}$ of the category Grp is such that

$$\mathcal{F}_{\mathsf{ex/reg}} = \mathsf{Grp}.$$

Groupoids and exact completion

Let $\mathsf{Eq}(\mathbb{C})$ be the category of equivalence relations in a semi-abelian category $\mathbb{C}$. It is a torsion-free subcategory of $\mathsf{Grpd}(\mathbb{C})$, where the torsion subcategory is $\mathsf{Ab}(\mathbb{C})$. Then :

$$[\mathsf{Eq}(\mathbb{C})]_{\mathsf{ex/reg}} = \mathsf{Grpd}(\mathbb{C}).$$

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This is not true when $\mathbb{C}$ is the category of sets.

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Remark

This is not true when $\mathbb{C}$ is the category of sets.

The category $\mathsf{NormMono}(\mathbb{C})$ of normal monomorphisms in a semi-abelian $\mathbb{C}$ is a semi-localization of a semi-abelian category, and

$$[\mathsf{NormMono}(\mathbb{C})]_{\mathsf{ex/reg}} = \mathsf{XMod}(\mathbb{C}),$$

where $\mathsf{XMod}(\mathbb{C})$ is the category of "internal crossed modules" (G. Janelidze, 2003).

Final remarks

- Internal groupoids in algebraic categories are important in commutator theory and universal algebra.

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- Internal groupoids in algebraic categories are important in commutator theory and universal algebra.
- They occur in homological and homotopical algebra also as crossed modules, crossed squares, and central extensions.
- $\mathrm{Grpd}(\mathbb{C})$ is a source of non-abelian torsion theories.
- $\mathrm{Grpd}(\mathbb{C})$ is the solution of a universal "exactness" problem.
- Internal groupoids have a central role in the fundamental theorem of categorical Galois theory.

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