$\mathbb{E}_k$-centers of monoidal ∞-categories

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Preliminaries

- Fin_* : Obj(Fin_*)$= \{\langle n \rangle = \{*, 1, \ldots, n\} : n \in \mathbb{N}\}$, Mor($\mathrm{Fin}_*$)$= \{f : \langle m \rangle \to \langle n \rangle : f(*) = *\}$
- For any (nonpointed) finite set S, $\mathrm{Rect}((0,1)^k \times S, (0,1)^k)$ $\rightsquigarrow$ the set of *rectilinear embeddings*, where a rectilinear embedding $r : (0,1)^k \times S \to (0,1)^k$ is
  - an open embedding,
  - whose restrictions to $(0,1)^k \times \{j\}$ are of the form $r(x,j) = D_j x + b_j$ where $D_j = diag(\{d_j^i\})$ and $d_j^i > 0$.

  This set will be endowed with the compact-open topology.
- $\mathcal{C}$ topological category $\rightsquigarrow$ category enriched in topological spaces.

  Nerve $N(\mathcal{C})$ $\rightsquigarrow$ an ∞-category, whose objects are the same of $\mathcal{C}$, and whose n-simplexes are functors

$$\{0 \to 1 \to \cdots \to n-1 \to n\} \to \mathcal{C}_\Delta.$$

The ∞-operad $\mathbb{E}_k$

Definition

Let ${}^t\mathbb{E}_k$ be the topological category with $\mathrm{Obj}(\mathrm{Fin}_*)$ as objects and as morphisms the pairs $(f, \{r_j\}_{j=1,\dots,n})$ where

- $f \in \mathrm{Hom}_{\mathrm{Fin}_*}(\langle m\rangle, \langle n\rangle)$,
- for any $j \in \{1, \dots, n\}$, $r_j : (0,1)^k \times f^{-1}(j) \longrightarrow (0,1)^k$ is a rectilinear embedding.

Topology on $\mathrm{Hom}_{{}^t\mathbb{E}_k}(\langle m\rangle, \langle n\rangle)$: induced by the topology on each $\mathrm{Rect}((0,1)^k \times f^{-1}(j), (0,1)^k)$.

We denote by $\mathbb{E}_k$ the nerve $N({}^t\mathbb{E}_k)$.

Remark

The forgetful functor ${}^t\mathbb{E}_k \to \mathrm{Fin}_*$ *induces a functor* $\mathbb{E}_k \to \mathrm{N}(\mathrm{Fin}_*)$ *which exhibits* $\mathbb{E}_k$ *as an* ∞*-operad.*

Fundamental Example: the loop space ΩX

Any loop space ΩX has an operation $\Omega X \times \Omega X \longrightarrow \Omega X$, given by concatenation of loops, associative **up to homotopy**, but **not** strictly associative.

$n \in \mathbb{N}, r : (0,1) \times \{1, \dots, n\} \to (0,1)$ rectilinear embedding $\rightsquigarrow$ well-defined n-ary concatenation law using r:

This defines a functor ${}^t\mathbb{E}_1 \longrightarrow \mathrm{Top}^\times$:

1. $\langle n \rangle \mapsto (\Omega X)^n$ on objects, in particular $\langle 1 \rangle \mapsto \Omega X$
2. on morphisms, $(\langle m \rangle \xrightarrow{f} \langle n \rangle, \{r_j\}) \mapsto (\Omega X)^m \longrightarrow (\Omega X)^n$ where $(\Omega X)^{f^{-1}(j)} \to \Omega X$ is the multiple concatenation induced by r_j.

$\mathbb{E}_k$-algebras in symmetric monoidal ∞-categories

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal (∞-)category (Spaces$^{\times}$, Cat$^{\times}$, Cat$_{\infty}^{\times}$).

Role of $\mathbb{E}_k \rightsquigarrow$ to encode the datum of k different multiplication laws on some $A \in \mathcal{C}^{\otimes}$, associative *up to homotopy*, that distribute with respect to one another *up to homotopy*.

Definition

An $\mathbb{E}_k$-**algebra** *in* $\mathcal{C}^{\otimes}$ *is a functor* $\phi : \mathbb{E}_k \to \mathcal{C}^{\otimes}$ *over* Fin$_*$ *s.t.*

(*) ϕ *sends inert morphisms to cocartesian morphisms.*

The object $A = \phi(\langle 1 \rangle)$ *is called the underlying object of* ϕ*. We say that* $\mathbb{E}_k$ **acts on** $A = \phi(\langle 1 \rangle)$*.*

Remark

ΩX *is an* $\mathbb{E}_1$*-algebra in* Spaces*. This generalizes to the case of k-fold loop spaces, exhibiting* $\Omega^k X$ *as an* $\mathbb{E}_k$*-algebra in* Spaces*.*

$\mathbb{E}_k$-categories

Definition

An $\mathbb{E}_k$-(∞-)category is an $\mathbb{E}_k$-algebra in (∞-)categories.

$\mathbb{E}_1$-category $\rightsquigarrow$ monoidal category (no nontrivial notion of homotopy in the mapping space of a category).

$\mathbb{E}_2$-category $\rightsquigarrow$ *braided* monoidal category (Eckmann-Hilton's theorem).

$\mathbb{E}_k$-category, $k \geq 3$ $\rightsquigarrow$ symmetric monoidal category (mapping spaces between categories are "too small" to contain nontrivial higher coherences).

An $\mathbb{E}_k$-∞-category, $k > 2$, has "stronger commutativities" than an $\mathbb{E}_2$-∞-category, but is not symmetric monoidal.

$\mathbb{E}_k$-centers of monoidal ∞-categories

There is a way to add structure to an $\mathbb{E}_k$-∞-category to build a "universal" associated $\mathbb{E}_{k+1}$-∞-category.

Definition

Let $\mathcal{C}$ be an $\mathbb{E}_k$-∞-category. A **center** *of $\mathcal{C}$ is a final object in the ∞-category of $\mathbb{E}_{k+1}$-algebras acting on $\mathcal{C}$.*

The center is unique up to equivalence and is denoted by $Z_{\mathbb{E}_k}(\mathcal{C})$.

Remark

$k = 1$, $\mathcal{C}$ ordinary category $\rightsquigarrow Z_{\mathbb{E}_1}(\mathcal{C})$ coincides with the classical notion of **Drinfeld center** *of a monoidal category.*

The classical Geometric Satake Theorem

Fix a complex reductive algebraic group G (e.g. GL_n). $\check{G}$ is the Langlands dual group ($\check{GL_n} \simeq GL_n$).

Theorem (Geometric Satake Theorem; Ginzburg, Mirkovic/Vilonen)

There is an equivalence

$$(\mathrm{Rep}_G, \otimes) \simeq (\mathrm{Perv}_{\check{G}_{\mathcal{O}}}(\mathrm{Gr}_{\check{G}}), \star)$$

of abelian symmetric monoidal categories.

We want to find a derived version of this theorem (motivation: the Geometric Langlands Conjecture is derived).

The Derived Satake Theorem

The derived ∞-category of representations DRep_G is an $\mathbb{E}_2$-∞-category (forget the $\mathbb{E}_\infty$-structure... strange, but makes sense at a derived level)
$\rightsquigarrow Z_{\mathbb{E}_2}(\mathrm{DRep}_G)$, $\mathbb{E}_3$-∞-category acting on DRep_G.

Theorem (Bezrukavnikov-Finkelberg (triangulated), Haine-Macerato-N. (homotopy-coherent, work in progress))

Analogue of Geometric Satake Equivalence:

$$Z_{\mathbb{E}_2}(\mathrm{DRep}_G) \simeq \mathrm{Sph}(\check{G})$$

as ∞-categories, where $\mathrm{Sph}(\check{G})$ is the natural derived analogue of $Perv_{\check{G}_{\mathcal{O}}}(Gr_{\check{G}})$.

[N. 2021]: construction of an "intrinsic" $\mathbb{E}_3$-structure on the right-hand-side, making the equivalence $\mathbb{E}_3$-monoidal (cfr. symmetric monoidality in the Geometric Satake Theorem).

Thank you for your attention!