A sketch of topological semi-Galois theory

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Classical Galois theory

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Subfield fixed by $H \leftarrow\!\shortmid H$

Classical Galois theory

This is a categorical equivalence!

If we instead identify a subgroup $H \leq G$ with the set of cosets G/H, equipped with left action by G, then we have G-set homomorphisms between these in correspondence with ring homomorphisms over k.

Classical Galois theory

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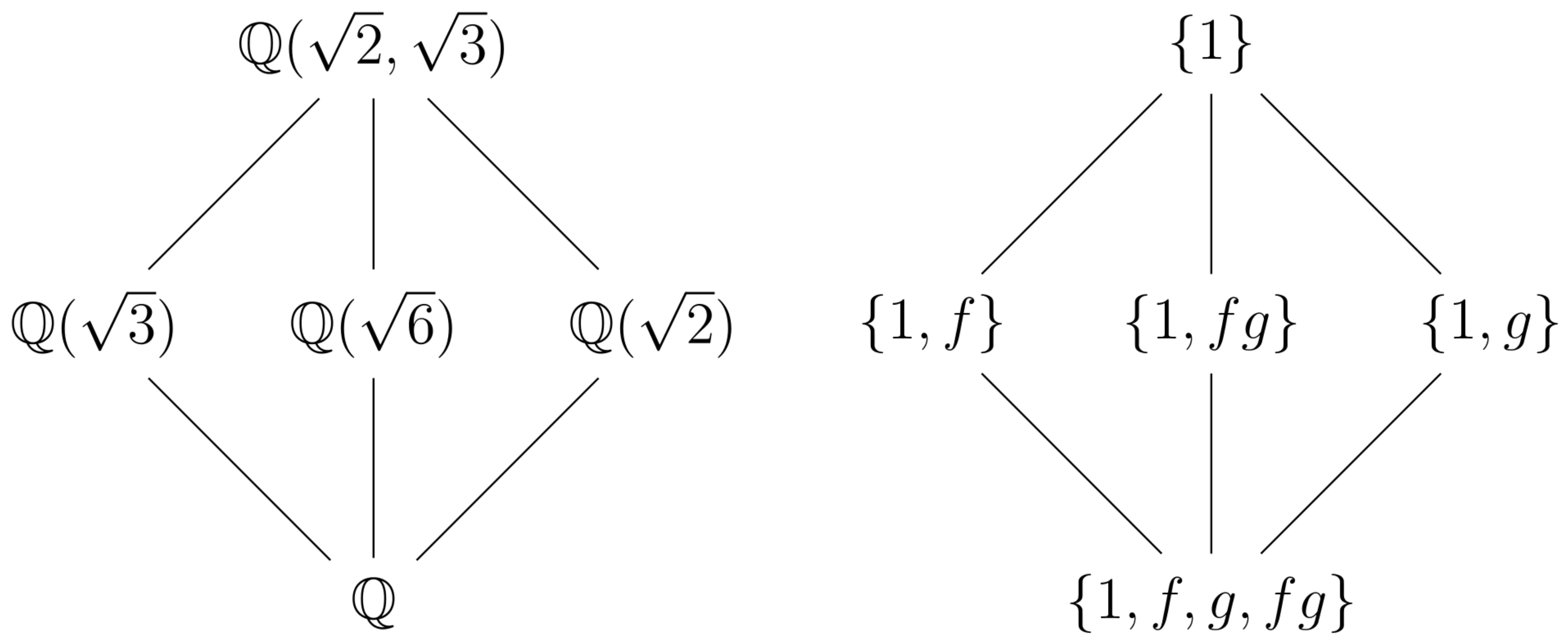

Classical Galois theory

Actions of
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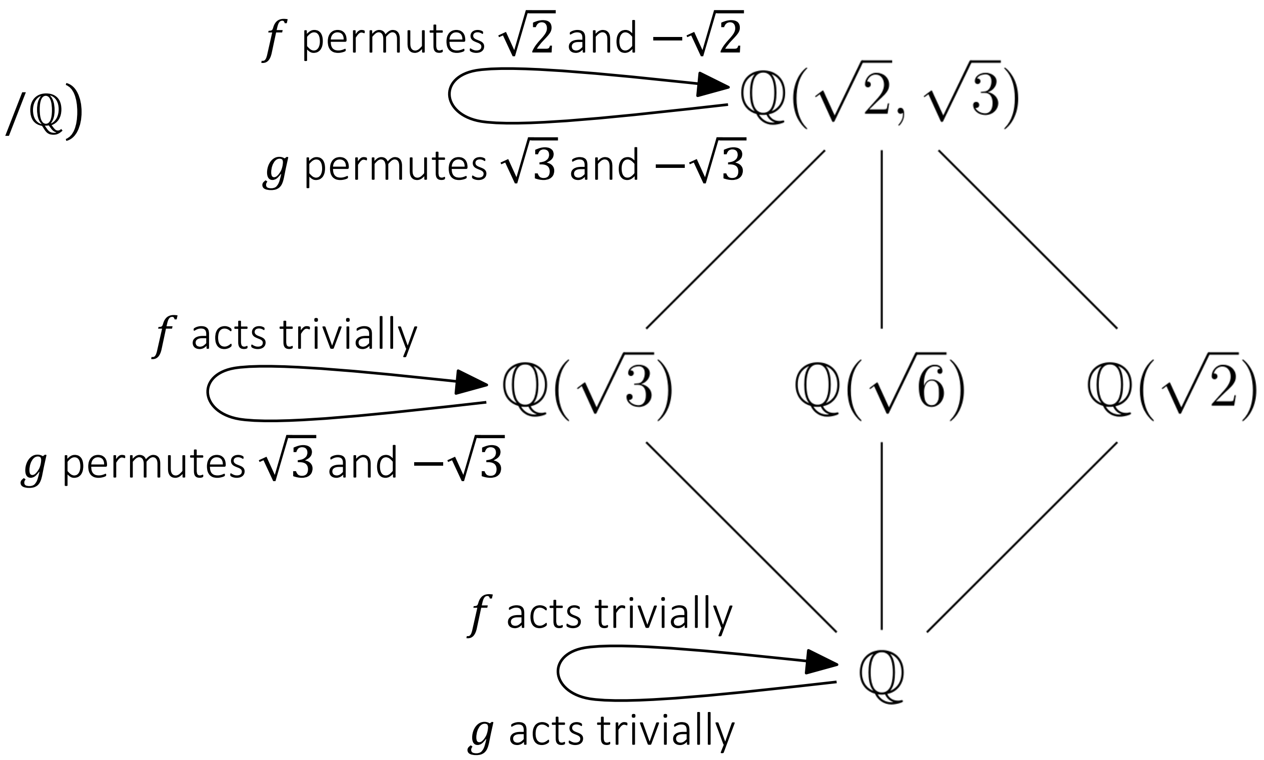

“semi-”Galois theory

The aim of **semi-Galois theory** is to replace groups with monoids, and identify the extent to which the left hand side can be extended as a result.

We do this through the medium of topos theory, extending Caramello’s development of *Topological Galois Theory* (TGT).

Topological Galois theory

A central result in Caramello's TGT is the identification of conditions on a category $\mathcal{C}$ in order to obtain an equivalence of the form $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}}) \simeq \mathbf{Cont}(G, \tau)$.

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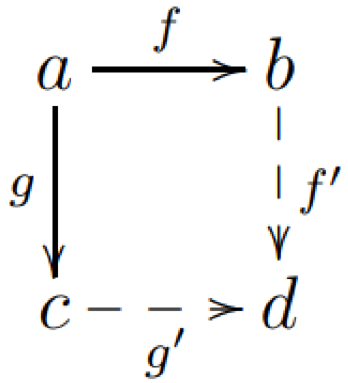

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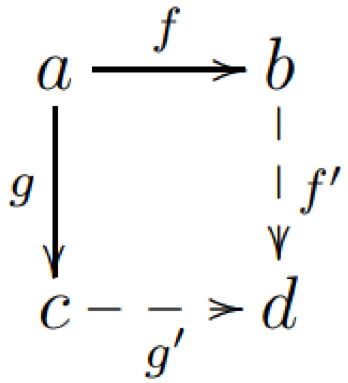

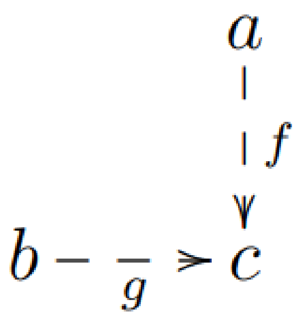

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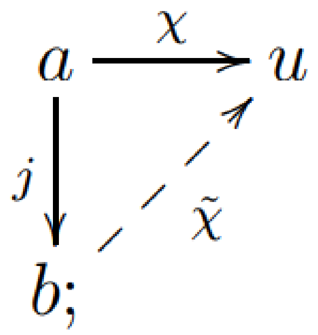

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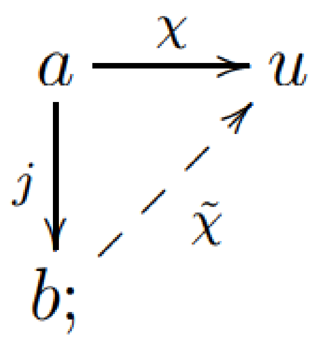

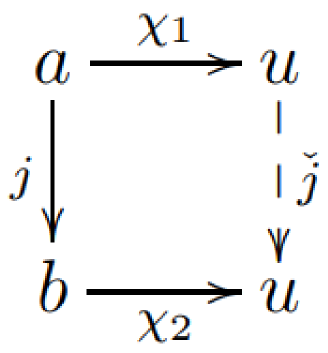

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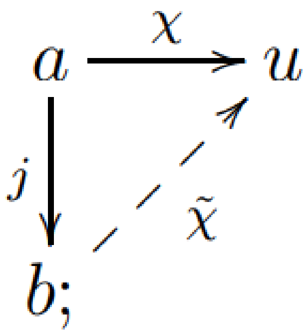
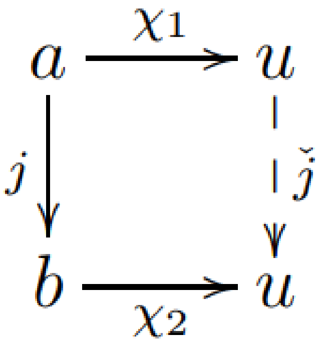
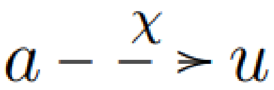
$\mathcal{C}$-**ultrahomogeneous** if every morphism in $\mathcal{C}$ extends to an automorphism of u,

$\mathcal{C}$-**universal** if every object of $\mathcal{C}$ admits a morphism to u.

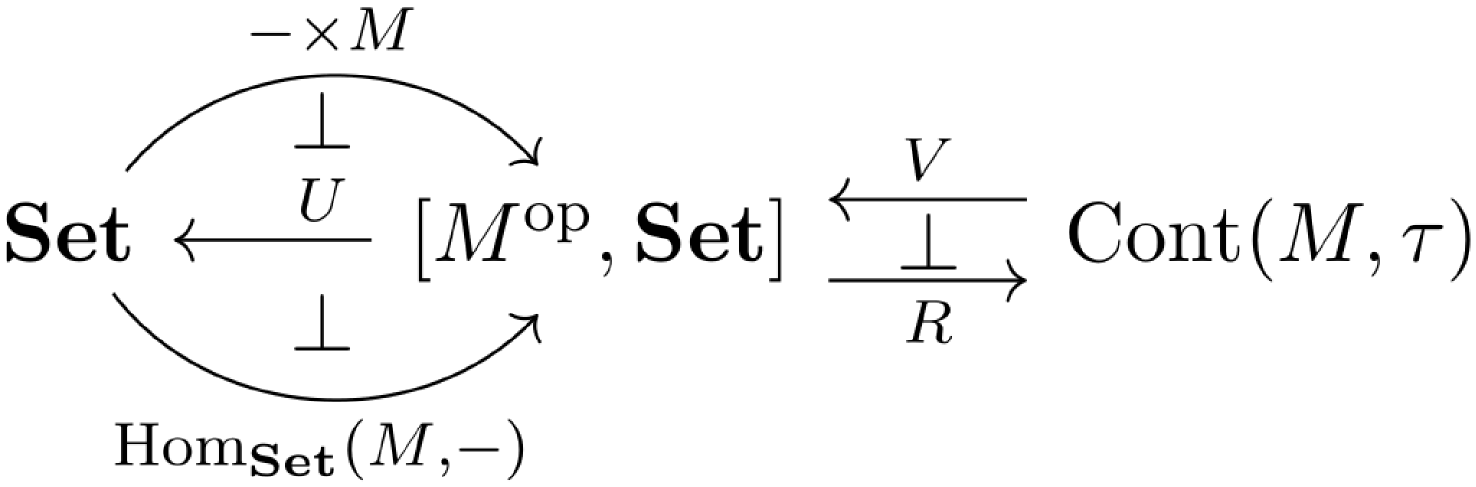






Where does this come from?

We can characterize toposes of topological monoid actions by the existence of a surjective point of a particular form:



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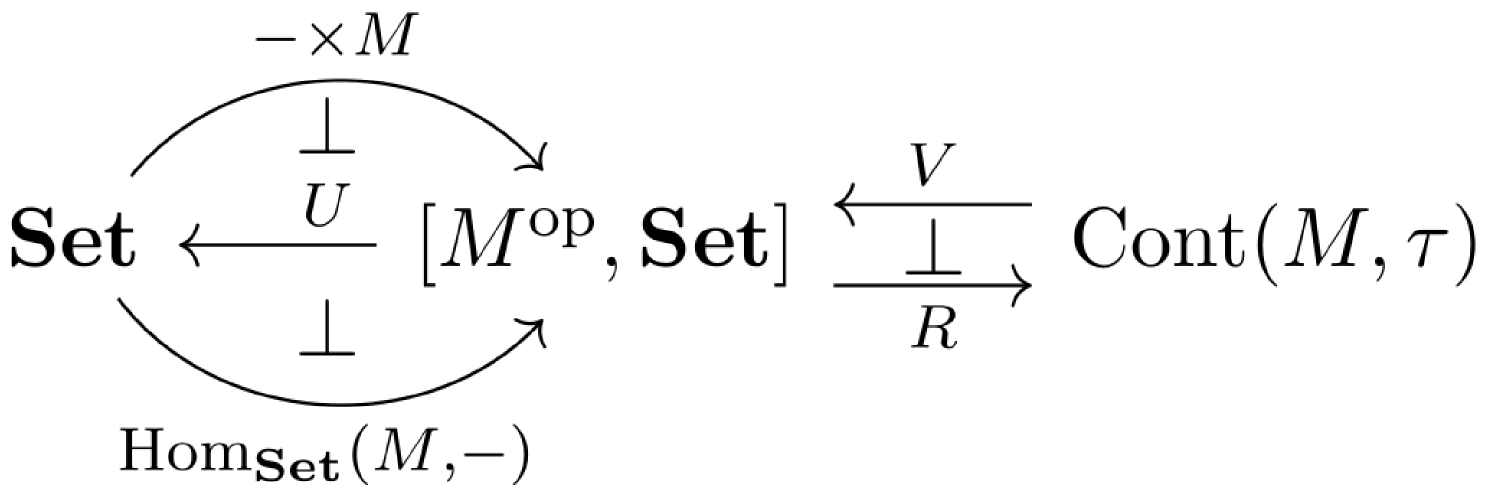

Groups correspond to the **atomic** case, so we consider atomic sites!

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On the other hand, any point of $\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ extends to a point of $\mathbf{PSh}(\mathcal{C}^{\mathrm{op}})$, and the latter correspond to objects of $\mathbf{Ind}(\mathcal{C})$. The conditions therefore translate to conditions for u to produce a point factoring through $\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$.

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The conditions on $\mathcal{C}$ are necessary conditions which under certain conditions (such as countability) are also sufficient, as we shall see later.

Principal sites

For the general case, we replace atomic sites with **principal sites**.

In such a site, any covering sieve must be reducible to (a sieve generated by) a single morphism. As such, we consider a class $\mathcal{T}$ of morphisms which will generate covering sieves.

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The conditions to follow are necessary and sufficient for $\mathcal{T}$-morphisms to be singleton presieves generating a Grothendieck topology, the **principal topology generated by** $\mathcal{T}$, $J_{\mathcal{T}}$, and such that moreover $\mathcal{T}$ is recoverable as the representable morphisms sent to epimorphisms.

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$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & D \end{array}$$

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4. Given any morphism f of $\mathcal{C}$ such that $f \circ g \in \mathcal{T}$ for some morphism g, then $f \in \mathcal{T}$.

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The analogue of the joint embedding property is the **joint $\mathcal{T}$-covering property**, where every pair of objects of $\mathcal{C}$ admits $\mathcal{T}$-morphisms from a common object.

Generalizing the Ind-conditions

To generalize the conditions on the object of $\mathbf{Ind}(\mathcal{C}^{\mathrm{op}})$, we restrict to the special case in which the stable class $\mathcal{T}$ is part of an **orthogonal factorization system** (OFS) $(\mathcal{T},\mathcal{M})$ on $\mathcal{C}$ such that $\mathcal{M}$ is contained in the class of monomorphisms, because this provides a convenient class of representable subobjects*.

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We can extend $(\mathcal{M}^{\mathrm{op}},\mathcal{T}^{\mathrm{op}})$ to an OFS $(\mathcal{L},\mathcal{R})$ on $\mathbf{Ind}(\mathcal{C}^{\mathrm{op}})$, and we use this to describe the conditions.

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$\mathcal{R}$**-universal** if every object of $\mathcal{C}$ admits an $\mathcal{R}$-morphism to u,

$\mathcal{T}$**-constructible** if u is expressible as a formal directed colimit of a diagram whose morphisms lie in $\mathcal{T}$.

Generalizing the Ind-conditions

The conclusion we sought is that if $\mathcal{C}$ is a category satisfying all of these constraints and there exists a $\mathcal{T}$-injective, $\mathcal{C}$-quasi-homogeneous, $\mathcal{R}$-universal, $\mathcal{T}$-constructible object u in $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$, then **the topologized endomorphism monoid of u provides a presentation of the topos $\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathcal{T}})$**.

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Moreover, if $\mathcal{C}$ is countable and satisfies the necessary conditions mentioned earlier, then we may directly construct such an object u.

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An equivalence $\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathcal{T}}) \simeq \mathrm{Cont}(M, \tau)$ gives us an (ana)functor $\mathcal{C} \to \mathfrak{R}_{\tau}$, where the latter is the category of open right congruences on (M, τ).

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3. Any pair of objects X, Y in $\mathcal{C}$ are jointly covered by some object Z via strict epimorphisms.
4. $\mathcal{C}$ is effectual*.

What Next?

I had hoped to have time to complete the work of translating the categorical conditions into algebraic constraints on algebras over a ring, but there is still work to be done!

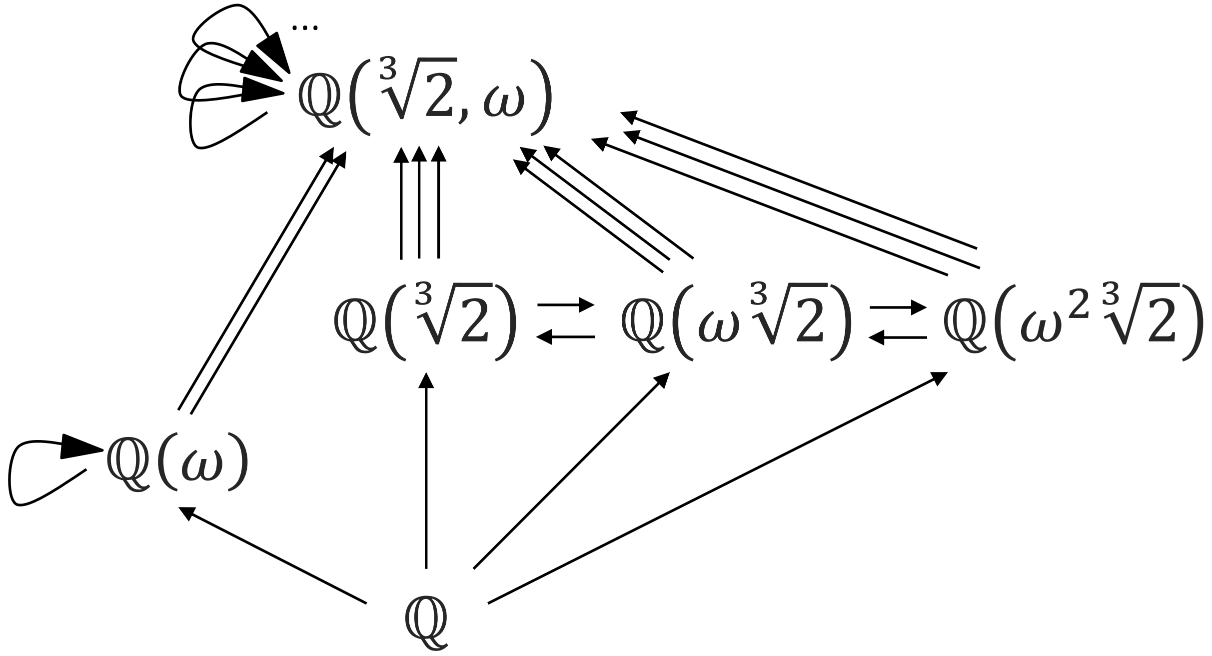
Thank You!

Effectual?

A reductive category is **effectual** if for every funneling diagram $F : \mathcal{D} \to \mathcal{C}$ with colimit expressed by $\lambda : F(D_0) \to C_0$, given morphisms $g_1, g_2 : C \rightrightarrows F(D_0)$ in $\mathcal{C}$ such that $\lambda \circ g_1 = \lambda \circ g_2$, there is a strict epimorphism $t : C' \to C$ such that $g_1 \circ t$ and $g_2 \circ t$ lie in the same connected component of $(C' \downarrow F)$.

Example

Consider the extension $\mathbb{Q} \leq \mathbb{Q}(\sqrt[3]{2}, \omega)$, where ω is a complex cube root of unity. This is a Galois extension with Galois group the six element dihedral group D_6, but several intermediate extensions are *not* Galois...



Normality guarantees that every morphism is a strict monomorphism, so that the atomic topology coincides with the reductive topology.

“Classical” semi-Galois theory

Let $R \hookrightarrow S$ be an injective (commutative) ring homomorphism.

We say this extension is *algebraic* if every element of S is a root of some polynomial in R. We can study the category of subrings of S which are finite extensions of R, and the homomorphisms between these which fix R.

For each element $\alpha \in S$, we may consider the ideal $I_\alpha \subseteq R[x]$ of polynomials satisfied by α, and in turn we may consider $[\alpha] \subseteq S$, which is the subset of elements which are roots of all polynomials in I_α. Any homomorphism must preserve this subset; indeed, $I_\alpha \subseteq I_{f(\alpha)}$ for any homomorphism f.